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Application of the Fractional Reduced Differential Transform Method to a Time-Fractional Heat Equation

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Abstract

This work deals with the numerical solution of a time-fractional heat equation where a Caputo fractional derivative of order $0< \alpha \leq 1$ is used in place of the traditional first-order time derivative. This change improves the model's capacity to represent anomalous diffusion behavior and memory effects, which are frequently seen in intricate engineering and physical systems. Applying and evaluating the Fractional Reduced Differential Transform Method (FRDTM) to solve this fractional-order partial differential equation is the aim of this work. The Fractional Variational Iteration Method (FVIM) was used to validate the findings. For different fractional orders, namely$\alpha = 0.2, 0.5, 0.7$, and the classical case where $\alpha = 1$ with a known exact solution, two numerical examples were performed. The findings demonstrate that FRDTM offers extremely stable and accurate solutions that closely match the exact solution in the classical case ($\alpha = 1$). When it comes to capturing the change from rapid decay at lower fractional orders to more sustained solution profiles as the order increases, the FRDTM performs better than the FVIM. The differences between the two methods demonstrate FRDTM's superior convergence and accuracy across all cases considered. Finally, this study demonstrates the effectiveness of FRDTM as reliable semi-analytical tool for solving fractional heat problems, and it contributes to advancing computational approaches for solving partial differential equations in science and engineering.

Keywords: Anomalous diffusion, FRDTM, FVIM, heat equation, fractional derivative

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Introduction

The heat equation, a fundamental partial differential equation, describes how temperature evolves in a region over time. It falls into the category of PDEs known as parabolic equations, which are distinguished by their connection to diffusion processes [1]. The heat equation has numerous applications in applied mathematics and engineering, particularly in industrial processes such as heat exchangers, distillation columns, and the diffusion of chemicals through various media [2, 3].

In Evans [4], the integer order heat equation in the domain $\Omega \subset \mathbb{R}^n$ $(n = 1,2)$ is expressed as follows;

$$u_t(x,t) = a\Delta u(x,t) + q(x,t), \quad x \in \Omega, \ t > 0 \qquad (1)$$

Subject to the initial condition and the homogeneous Dirichlet boundary condition

$$u(x,0) = f(x) \qquad (2a)$$
$$u(x,t) = g(t) \qquad (2b)$$

where Δu is the Laplacian operator expressed as,

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^n u}{\partial {x_i}^n} \qquad (3)$$

And $u(x,t)$ is the temperature at a specific spatial point $x = (x_1, x_2, x_3, .., x_n)$,t is the time variable, $a > 0$ is thermal diffusivity, and $q(x,t)$ is source term;the equation is considered homogeneous when the source term $q(x,t)$equals zero. Although exact solutions of such equations exist under certain conditions, many real-life processes exhibit memory and nonlocal effects that are not adequately captured by integer-order models. To address this limitation, fractional derivatives have been introduced as powerful tools for modeling anomalous diffusion and other complex dynamics in heterogeneous or memory-dependent media [5, 6].

Fractional-order models, especially those using the Caputo derivative, are particularly good at describing systems with sub-diffusion or super-diffusion characteristics. These models generalize integer-order equations by replacing the standard first-order time derivative with a fractional derivative of order, where this change enables the model to account for long-term memory effects in the system [7]. Metzler and Klafter [5] employed the fractional Fokker–Planck equation to model sub-diffusive phenomena occurring near thermal equilibrium. Yuste and Lindenberg [8] employed a fractional diffusion equation to model the sub-diffusion of particles exhibiting coagulation and annihilation dynamics. Chang and Sun [9] explored the gas transport process in heterogeneous media using a fractional

advection–dispersion model, which can characterize heavy-tailed behavior and early breakthrough phenomena in transport. Ahmad *et al*. [10] investigated a three-dimensional multi-term fractional anomalous solute transport model for groundwater contamination.
The nonlocal nature of fractional derivatives often makes analytical solutions difficult to obtain. Consequently, numerical methods have become essential tools for solving fractional diffusion equations. Several approaches have been employed by researchers, including the Fractional Variational Iteration Method [11], the Fractional Homotopy Analysis Method [12], the Fractional Homotopy Perturbation Method [13], the Generalized Fractional Differential Transform Method [14], the Fractional Sumudu Decomposition Method [15], the Radial Basis Function Finite Difference Method [16], and the Chebyshev Collocation Method [17]. Among these, the Fractional Reduced Differential Transform Method (FRDTM) has attracted considerable attention for its simplicity, efficiency, and accuracy [18]. In this study, we aim to apply the FRDTM to solve time-fractional heat equations. The results obtain by FRDTM will be compared with the Fractional Variational Iteration Method (FVIM) and validated against exact solutions. We will also analyze the system's behavior by varying fractional order α.

Materials and Methods

The Fractional Reduced Differential Transform Method (FRDTM)

The basic definitions of the Fractional Reduced Differential Transform Method (FRDTM) and its inverse transform are presented in [19] as follows:
Let $u(x,t)$ be a function of two variables expressed as product of functions as,

$$u(x,t) = v(x)w(t) \qquad (4)$$

Based on the properties of the Fractional Differential Transform Method (FDTM) [20], we have:

$$u(x,t) = \sum_{i=0}^{\infty} v(i)x^i \times \sum_{j=0}^{\infty} w(j)t^j = \sum_{k=0}^{\infty} U_k(x)t^k \qquad (5)$$

where U_k is the t-dimensional spectrum function of $u(x,t)$

Definition 1: If the function $u(x,t)$ is analytical and k-times continuously differentiable with α^{th} derivatives with respect to the time t and space x in the domain of interest, then let the Fractional Reduced Differential Transform (FRDT) of $u(x,t)$ be given as

$$\mathrm{FRDT}[u(x,t)] = V_k(x) = \frac{1}{\Gamma(k\alpha+1)}\left[(D_{a,t}^{\alpha})^k(u(x,t)\right]_{t=t_0} = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=t_0}, \quad k = 0,1.2,\ldots \qquad (6)$$

Where $\alpha > 0$ the time fractional order derivative, t-dimensional spectrum function $U_k(x)$ is the transformation function and $u(x,t)$ is the original function.

Definition 2: The differential inverse fractional reduced transform of $U_k(x)$ denoted by $u(x,t)$ is given by

$$\mathrm{FRDT}^{-1}(U_k(x)) = u(x,t) = \sum_{k=0}^{\infty} U_k(x)(t-t_0)^{k\alpha} \qquad (7)$$

Combining equations (6) and (7), we have

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{k\alpha}}{\partial t^{k\alpha}}u(x,t)\right]_{t=t_0}(t-t_0)^{k\alpha} \qquad (8)$$

Equation (8) clearly shows that the concept of the reduced differential transform is derived from the power series expansion. Since the initial conditions are expressed in terms of integer-order derivatives, the corresponding transformations of the initial conditions are defined as:

$$U_k = \begin{cases} \dfrac{1}{(k/\alpha)!}\left[\dfrac{\partial^{\frac{k}{\alpha}}}{\partial t^{\frac{k}{\alpha}}}u(t)\right]_{t=t_0} & \text{for } k = 0,1,2,3,..(m\alpha-1) \text{ if } k/\alpha \in \mathbb{Z}^+ \\ 0 & \text{if } k/\alpha \notin \mathbb{Z}^+ \end{cases} \qquad (9)$$

Theorems of Fractional Reduced Differential Transform Method (FRDTM)

We present some basic theorems of the Fractional Reduced Differential Transform Method (FRDTM) as explained in a study [21]. Let $u(x,t))$, $v(x,t))$, and $w(x,t)$ be analytical and k-times continuously differentiable functions with respect to the space variable x and the time variable t. Then, the following theorems are given below:

Theorem 1: if $u(x,t) = v(x,t)$ then $U_k(x,t) = V_k(x,t)$

Theorem 2: if $u(x,t) = v(x,t) \pm w(x,t)$, then $U_k(x,t) = V_k(x,t) \pm W_k(x,t)$

Theorem 3: if $u(x,t) = cv(x,t)$, where c is a constant, then $U_k(x,t)= cV_k(x,t)$

Theorem 4: if $u(x,t) = \frac{\partial^n}{\partial x^n}v(x,t)$, then $U_k(x,t) = \frac{\partial^n}{\partial x^n}V_k(x,t)$.

Theorem 5: if $v(x,t) = \frac{\partial^{\alpha n}}{\partial t^{\alpha n}}u(x,t)$, then $V_k(x,t) = \frac{\Gamma(\alpha(k+n)+1)}{\Gamma(k\alpha+1)}U_k(k+n)$, $n = 1,2,3,\ldots$

Theorem 6: if $u(x,t) = v(x,t)w(x,t)$, then $U_k = \sum_{i=0}^{k} V_i(x)W_{k-i}(x)$.

Fractional Variational Iterational Method (FVIM) for validation

To evaluate the accuracy of the FRDTM, we employ the Fractional Variational Iteration Method (FVIM). FVIM is a semi-analytical technique used to solve Fractional Differential Equations (FDEs), including ordinary, partial, and integro-differential types. It extends the Variational Iteration Method (VIM), originally proposed by Ji-Huan He in 1999 [22], to

fractional-order systems. In this study, FVIM serves as a reference approach for comparison.
The FVIM correction functional for equation (1) is constructed as follows [23]:

$$u_{k+1}(x,t) = u_k(x) + \frac{1}{\Gamma(\alpha+1)}\int_0^t \lambda(\tau)[u_\tau(x,\tau)\text{-}a\Delta u_k(x,\tau)+q(x,\tau)](d\tau)^\alpha \qquad (10)$$

where $0 < \alpha \leq 1$, $\lambda(\tau) = -1$ is the Lagrange multiplier and u_{k+1} is the $(k+1)$th approximation produced by the variational iteration correction functional. For information on the derivation and applications of FVIM, see Odibat and Momani [24].

Numerical Examples

Problem 1: Consider the time-fractional one-dimensional equation

$$D_t^\alpha = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = 3\frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < \alpha \leq 1, \qquad (11)$$

Subject to the initial and boundary conditions

$$u(x,0) = x, \qquad (12a)$$
$$u(0,t) = 0 \qquad (12b)$$
$$u(2,t) = 0 \qquad (12c)$$

With the exact solution given as

$$u(x,t) = \sum_{k=1}^{\infty} \frac{4}{k\pi}(-1)^{k+1}\sin\left(\frac{k\pi x}{2}\right)E_\alpha\left(-\frac{3\pi^2 k^2}{4}t^\alpha\right) \qquad (13)$$

where D_t^α is the Caputo fractional derivative operator defined as [24]

And $u(x,t)$ represents the distribution of heat over space x and time t, $\Gamma(.)$ is the gamma function. Since the exact solution is expressed in terms of sine functions, the initial condition $u(x,0) = x$ over the interval $[0,2]$ is expressed using a Fourier sine series. The sine series expansion is given as:

$$u(x,0) = \sum_{k=1}^{\infty} \frac{4}{k\pi}(-1)^{k+1}\sin\left(\frac{k\pi x}{2}\right) \qquad (15)$$

Implementation of Fractional Reduced Differential Transform Method (FRDTM) on problem 1

In this section, weapply the Fractional Reduced Differential Transform operator FRDT on both sides of equation (11) and (12a)
By using Theorem 4 and 5, equation (11) is transformed to

$$U_{k+1}(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k+1)+1)}\left(3\frac{\partial^2 U_k(x)}{\partial x^2}\right) \qquad (16)$$

The initial condition and in equation (12a) can be transformed by using equation (9) to obtain

$$U_0(x) = \sum_{k=1}^{\infty} \frac{4}{k\pi}(-1)^{k+1}\sin\left(\frac{k\pi x}{2}\right) \qquad (17)$$

using equation (16) in (17), we obtained the following $U_k(x)$ values successively for $k = 0,1,2,3,4,\ldots$,

for $k = 0$

$$D_t^\alpha = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{\frac{\partial u(\tau,t)}{\partial \tau})}{(t-\tau)^{\alpha-n+1}}d\tau, \quad 0 < \alpha \leq 1 \qquad (14)$$

$$U_{0+1}(x) = \frac{\Gamma(0\times\alpha+1)}{\Gamma(\alpha(0+1)+1)}\left(3\frac{\partial^2 U_0(x)}{\partial x^2}\right)$$

$$U_1(x) = \frac{\Gamma(1)}{\Gamma(\alpha(0+1)+1)}\left(3\frac{\partial^2}{\partial x^2}\left(\sum_{k=1}^{\infty}\frac{4}{k\pi}(-1)^{k+1}\sin\left(\frac{k\pi x}{2}\right)\right)\right)$$

$$U_1(x) = \frac{3}{\Gamma(\alpha+1)}\left(-\pi\sin\left(\frac{\pi x}{2}\right) - 3\pi\sin\left(\frac{3\pi x}{2}\right) - 5\pi\sin\left(\frac{5\pi x}{2}\right) - 7\pi\sin\left(\frac{7\pi x}{2}\right)\right)$$

for $k = 1$

$$U_{1+1}(x) = \frac{\Gamma(1\times\alpha+1)}{\Gamma(\alpha(1+1)+1)}\left(3\frac{\partial^2 U_1(x)}{\partial x^2}\right)$$

$$U_2(x) = \frac{\Gamma(1)}{\Gamma(\alpha(1+1)+1)}\left(3\frac{\partial^2}{\partial x^2}\left(\frac{3}{\Gamma(\alpha+1)}\left(-\pi\sin\left(\frac{\pi x}{2}\right) - 3\pi\sin\left(\frac{3\pi x}{2}\right) - 5\pi\sin\left(\frac{5\pi x}{2}\right) - 7\pi\sin\left(\frac{7\pi x}{2}\right)\right)\right)\right)$$

$$U_2(x) = \frac{9}{4\Gamma(2\alpha+1)}\left(\pi^3\sin\left(\frac{\pi x}{2}\right) + 27\pi^3\sin\left(\frac{3\pi x}{2}\right) + 125\pi^3\sin\left(\frac{5\pi x}{2}\right) + 343\pi^3\sin\left(\frac{7\pi x}{2}\right)\right)$$

Then, using the inverse transformation rule in equation (7), the approximate solution of $u(x,t)$ is obtained by

$$u(x,t) = \sum_{k=0}^{10} U_k(x)t^{k\alpha}, \quad k = 0,\ 1,\ 2,\ 3,\ 4 \ \text{and}\ \alpha = 0.2, 0.5, 0.7 \text{ and } 1 \qquad (18)$$

$$u(x,t) = U_0(x) + U_1(x)t^{\alpha} + U_2(x)t^{2\alpha} + \cdots$$

$$u(x,t) = \left(\frac{4}{\pi}\sin\left(\frac{\pi x}{2}\right) + \frac{4}{3\pi}\sin\left(\frac{3\pi x}{2}\right) + \frac{4}{5\pi}\pi\sin\left(\frac{5\pi x}{2}\right) + \frac{4}{7\pi}\pi\sin\left(\frac{7\pi x}{2}\right) + \frac{3}{\Gamma(\alpha+1)}\right)\left(-\pi\sin\left(\frac{\pi x}{2}\right) - 3\pi\sin\left(\frac{3\pi x}{2}\right) - 5\pi\sin\left(\frac{5\pi x}{2}\right) - 7\pi\sin\left(\frac{7\pi x}{2}\right)\right)t^{\alpha} + \frac{9}{4\Gamma(2\alpha+1)}\left(\pi^3\sin\left(\frac{\pi x}{2}\right) + 27\pi^3\sin\left(\frac{3\pi x}{2}\right) + 125\pi^3\sin\left(\frac{5\pi x}{2}\right) + 343\pi^3\sin\left(\frac{7\pi x}{2}\right)\right)t^{2\alpha} + \cdots$$

The Fractional Variational Iteration correction functional for equation (10) is constructed as

$$u_{k+1}(x,t) = u_k(x) + \frac{1}{\Gamma(\alpha+1)}\int_0^t \lambda(\tau)\left[\frac{\partial^{\alpha} u_k(x,\tau)}{\partial \tau^2} - 3\frac{\partial^2 u_k(x,\tau)}{\partial x^2}\right](d\tau)^{\alpha}, 0 < \alpha \le 1 \quad (19)$$

where $\lambda(\tau) = -1$ is the Lagrange multiplier.

Problem 2: Consider the time-fractional 2-dimensional heat equation.

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \qquad 0 < \alpha \le 1,\ \ 0 < x < \pi, and\ 0 < y < \pi, t > 0 \qquad (20)$$

Subject to the initial condition and boundary condition

$$u(x,y,0) = \sin(x)\sin(y) \qquad (21a)$$

$$u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0 \quad (21b)$$

With the exact solution is given as

$$u(x,y,t) = E_{\alpha}(-2t^{\alpha})\sin(x)\sin(y) \qquad (22)$$

where $E_{\alpha}(t)$ is One-parameter Mittag-Leffler function [25] defined as

$$E_{\alpha}(t) = \sum_{k=0}^{\infty}\frac{t^{\alpha}}{\Gamma(k\alpha+1)} \qquad (23)$$

Implementation of Fractional Reduced Differential Transform Method (FRDTM) on problem 2

In this section, weapply the Fractional Reduced Differential Transform operator FRDT on both sides of equation (20) and (21a)

By using Theorem 4 and 5, equation (20) is transformed to

$$U_{k+1}(x,y) = \frac{\Gamma(k\alpha+1)}{\Gamma(\alpha(k+1)+1)}\left(\frac{\partial^2 U_k(x,y)}{\partial x^2} + \frac{\partial^2 U_k(x,y)}{\partial y^2}\right) \qquad (24)$$

The initial condition in equation $(21a)$ can be transformed by using equation (9) to obtain

$$U_0 = \sin(x)\sin(y) \qquad (25)$$

Using equation (24) in (25), we obtained the following $U_k(x,y)$ values successively for $k = 0,1,2,3,4, \ldots$,

for $k = 0$

$$U_{0+1}(x,y) = \frac{\Gamma(0\times\alpha+1)}{\Gamma(\alpha(0+1)+1)}\left(\frac{\partial^2 U_0(x,y)}{\partial x^2} + \frac{\partial^2 U_0(x,y)}{\partial y^2}\right)$$

$$U_1(x,y) = \frac{\Gamma(1)}{\Gamma(\alpha(0+1)+1)}\left(\frac{\partial^2}{\partial x^2}(\sin(x)\sin(y)) + \frac{\partial^2}{\partial y^2}(\sin(x)\sin(y))\right)$$

$$U_1(x,t) = -\frac{2\sin(x)\sin(y)}{\Gamma(\alpha+1)}$$

for $k = 1$

$$U_{1+1}(x,y) = \frac{\Gamma(1\times\alpha+1)}{\Gamma(\alpha(1+1)+1)}\left(\frac{\partial^2 U_1(x,y)}{\partial x^2} + \frac{\partial^2 U_1(x,y)}{\partial y^2}\right)$$

$$U_{1+1}(x,y) = \frac{\Gamma(1\times\alpha+1)}{\Gamma(\alpha(1+1)+1)}\left(\frac{\partial^2}{\partial x^2}\left(\frac{-2\sin(x)\sin(y)}{\Gamma(\alpha+1)}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{-2\sin(x)\sin(y)}{\Gamma(\alpha+1)}\right)\right)$$

$$U_2(x,y) = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\left(\frac{1}{\Gamma(\alpha+1)}(4\sin(x)\sin(y))\right)$$

$$U_2(x,y) = \frac{4\sin(x)\sin(y)}{\Gamma(2\alpha+1)}$$

for $k = 2$

$$U_{1+2}(x,y) = \frac{\Gamma(1\times\alpha+1)}{\Gamma(\alpha(1+1)+1)}\left(\frac{\partial^2 U_2(x,y)}{\partial x^2} + \frac{\partial^2 U_2(x,y)}{\partial y^2}\right)$$

$$U_{2+1}(x,y) = \frac{\Gamma(2\times\alpha+1)}{\Gamma(\alpha(2+1)+1)}\left(\frac{\partial^2}{\partial x^2}(4\sin(x)\sin(y)) + \frac{\partial^2}{\partial y^2}(4\sin(x)\sin(y))\right)$$

$$U_3(x,y,t) = \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\left(\frac{1}{\Gamma(2\alpha+1)}(4\sin(x)\sin(y))\right)$$

$$U_3(x,y) = -\frac{8\sin(x)\sin(y)}{\Gamma(3\alpha+1)}$$

$$\vdots$$

Then, using the inverse transformation rule in equation (7), the approximate solution of $u(x,y,t)$ is obtained by

$$u(x,y,t) = \sum_{k=0}^{10} U_k(x,y)t^{k\alpha}, \quad k = 0,\ 1,\ 2,\ 3,\ 4 \text{ and } \alpha = 0.5, 0.7 \text{ and } 1 \qquad (26)$$

$$u(x,y,t) = U_0(x,y) + U_1(x,y)t^{\alpha} + U_2(x,y)t^{2\alpha} + U_3(x,y)t^{3\alpha} \ldots$$

$$u(x,y,t) = \sin(x)\sin(y) - \frac{2\sin(x)\sin(y)}{\Gamma(\alpha+1)}t^{\alpha} + \frac{4\sin(x)\sin(y)}{\Gamma(2\alpha+1)}t^{2\alpha} + -\frac{8\sin(x)\sin(y)}{\Gamma(3\alpha+1)}t^{3\alpha} + \cdots$$

The Fractional Variational Iteration (FVIM) correction functional for equation (20) is constructed as

$$u_{k+1}(x,y,t) = u_k(x,y) + \frac{1}{\Gamma(\alpha+1)}\int_0^t \lambda(\tau)\left[\frac{\partial^{\alpha} u(x,y,\tau)}{\partial\tau^{\alpha}} - \frac{\partial^2 u(x,y,\tau)}{\partial x^2} + \frac{\partial^2 u(x,y,\tau)}{\partial y^2}\right](d\tau)^{\alpha} \qquad (27)$$

where $\lambda(\tau) = -1$ is the Lagrange multiplier

Results and Discussion

In this section, the approximate solutions of the fractional heat equation in both one- and two-dimensional cases are compared.

For the one-dimensional case, approximate solutions were obtained for fractional orders $\alpha = 0.2, 0.5, 0.7$, and 1.0 using the Fractional Reduced Differential Transform Method (FRDTM) and the Fractional Variational Iteration Method (FVIM). The solutions were computed over the time interval $t \in [0,1]$ at a fixed spatial point $x = 0.3$, with the integer-order case corresponding to $\alpha = 1.0$.

For the two-dimensional case, the heat equation was solved using the same numerical approaches (FRDTM and FVIM) for fractional orders $\alpha = 0.5, 0.7$ and 1.0. The solutions were computed over the same time interval $t \in [0,1]$ at fixed spatial points $x = \frac{\pi}{5}$ and $y = \frac{\pi}{2}$.

All computations were carried out using Maple 21. The FRDTM was implemented up to the tenth iteration, while the FVIM was applied up to the tenth term, yielding approximate solutions for $u(x,t)$ and $u(x,y,t)$.

Table 1 show that both FRDTM and FVIM perfectly reproduce the classical solution of the model when the fractional order $\alpha = 1$ at $0 < t < 1$ and $0 < x < 2$. For all solutions $u(x,t)$ points, the approximate solutions match the exact solution very well, with absolute errors effectively zero.

Table 1: Numerical results for Problem 1 ($\alpha = 1.0$, $0 < x < 2$ and $t \in [0, 1]$)

x	t	Exact solution for $\alpha = 1$	FRDTM	FVIM	Absolute error $u(x,t)$ for $\alpha = 1$ $\lvert Exac - FRDTM\rvert$	Absolute error $u(x,t)$ for $\alpha = 1$ $\lvert Exact - FVIM\rvert$
0	0	0	0	0	0	0
0.1	0.05	0.06906582	0.06906582	0.06906582	0	0
0.2	0.1	0.04770088	0.04770088	0.04770088	0	0
0.3	0.15	0.03294501	0.03294501	0.03294501	0	0
0.4	0.2	0.02275374	0.02275374	0.02275374	0	0
0.5	0.25	0.01571506	0.01571506	0.01571506	0	0
0.6	0.3	0.01085373	0.01085373	0.01085373	0	0
0.7	0.35	0.00749622	0.00749622	0.00749622	0	0
0.8	0.4	0.00517733	0.00517733	0.00517733	0	0
0.9	0.45	0.00357576	0.00357576	0.00357576	0	0
1	0.5	0.00246963	0.00246963	0.00246963	0	0
1.1	0.55	0.00170567	0.00170567	0.00170567	0	0
1.2	0.6	0.00117804	0.00117804	0.00117804	0	0
1.3	0.65	0.00081362	0.00081362	0.00081362	0	0
1.4	0.7	0.00056193	0.00056193	0.00056193	0	0
1.5	0.75	0.0003881	0.0003881	0.0003881	0	0
1.6	0.8	0.00026805	0.00026805	0.00026805	0	0
1.7	0.85	0.00018513	0.00018513	0.00018513	0	0
1.8	0.9	0.00012786	0.00012786	0.00012786	0	0
1.9	0.95	8.83E-05	8.83E-05	8.83E-05	0	0
2.0	1.0	6.10E-05	6.10E-05	6.10E-05	0	0

Table 2 present the FRDTM and FVIM for $\alpha = 0.2$at $0 < t < 1$ and fixed $x = 0.3$. It was observed that the FRDTM method consistently yielded slightly higher solution values than FVIM, particularly at early times between $t = 0$ to 0.3. The difference diminished as time increased, reflecting the damping nature of the fractional system. Fig. 1 shows 2D surface plot of the result obtained by the FRDTM and the FVIM approximate solutions $u(x, t)$.

Table 2: Numerical results for Problem 1 ($\alpha = 0.2$,$x = 0.3$and $t \in [0, 1]$)

x	t	FRDTM	FVIM
0.3	0	0.276000000	0.264000000
	0.05	0.131654430	0.125930324
	0.1	0.090928217	0.086974816
	0.15	0.062800322	0.060069873
	0.2	0.043373560	0.041487753
	0.25	0.029956307	0.028653859
	0.3	0.020689570	0.019790024
	0.35	0.014289422	0.013668143
	0.4	0.009869107	0.009440015
	0.45	0.006816180	0.006519824
	0.5	0.004707651	0.004502970
	0.55	0.003251378	0.003110014
	0.6	0.002245591	0.002147957
	0.65	0.001550936	0.001483504
	0.7	0.000739810	0.000707644
	0.75	0.000510956	0.000488740
	0.8	0.000352896	0.000337553
	0.85	0.00024373	0.000233133
	0.9	0.000168334	0.000161016
	0.95	0.000243730	0.000233133
	1.0	0.131654430	0.125930324

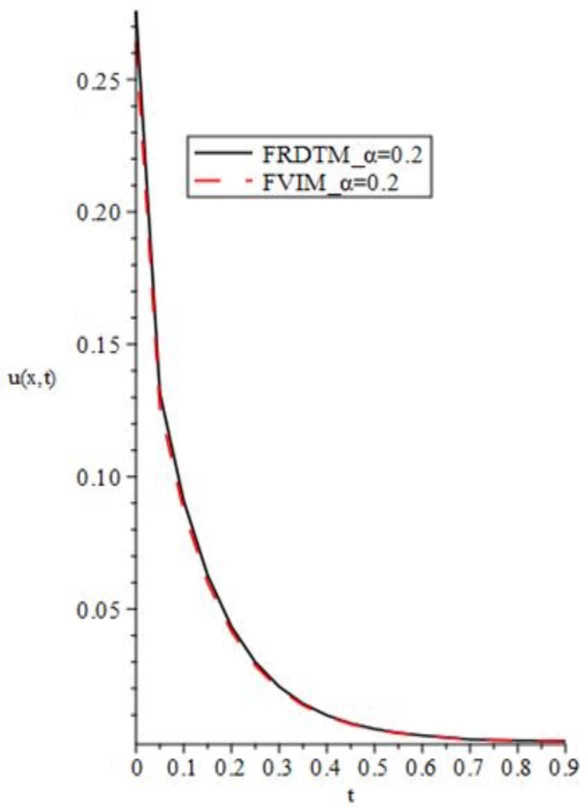

Figure 1: 2D surface plot comparing FRDTM and the FVIM approximate solutions $u(x, t)$ for $\alpha = 0.2$ at $0 < t \le 1$, $and\ x = 0.3$

Table 3: Numerical results for Problem 1 ($\alpha = 0.5$, $x = 0.3$ and $t \in [0, 1]$).

x	t	FRDTM	FVIM
0.3	0.0	0.277500000	0.015000000
	0.05	0.191657662	0.010359874
	0.1	0.132369943	0.007155132
	0.15	0.091422392	0.004941751
	0.2	0.043609286	0.002357259
	0.25	0.030119113	0.00162806
	0.3	0.020802013	0.001124433
	0.35	0.014367082	0.000776599
	0.4	0.009922744	0.000536365
	0.45	0.004733236	0.000255851
	0.5	0.003269048	0.000176705
	0.55	0.001559365	8.43E-05
	0.6	0.000743831	4.02E-05
	0.65	0.000513733	2.78E-05
	0.7	0.000354814	1.92E-05
	0.75	0.000245055	1.32E-05
	0.8	0.000169249	9.15E-06
	0.85	0.000121468	0.000118271
.	0.9	8.39E-05	8.17E-05
	0.95	5.79E-05	5.64E-05
	1	0.906317	0.882466

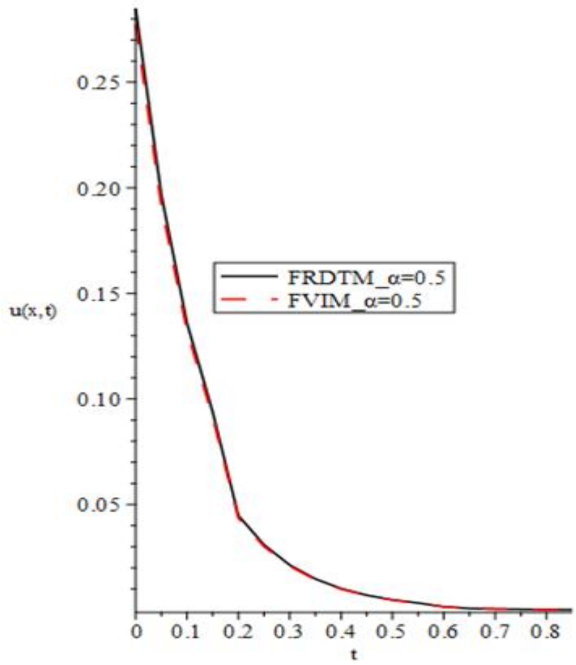

Figure 2: 2D surface plot comparing FRDTM and the FVIM approximate solutions $u(x, t)$ for $\alpha = 0.5$ at $0 < t \leq 1$ and $x = 0.3$.

Table 3 shows the results of FRDTM and FVIM for $\alpha = 0.5$, at $0 < t < 1$ and fixed $x = 0.3$, the numerical results reveal a significant discrepancy between FRDTM and FVIM, particularly in the early-time region. The FRDTM method begins with a much higher and more physically realistic value, whereas FVIM substantially underestimates the solution. As time increases, the two methods converge, with differences, dropping to negligible levels. Fig. 2 shows the 2D surface plot comparison between the FRDTM and FVIM solutions of $u(x, t)$.

Table 4 and Fig. 3 presents the result obtained by the FRDTM and the FVIM approximate solutions $u(x, t)$ for $\alpha = 0.7$ at fixed $x = 0.3$ and $0 < t < 1$. It was observed that both FRDTM and FVIM produce increasing solutions at $x = 0.3$ over time. The FRDTM results are consistently higher than those from FVIM across the entire time domain. While the difference is small at early times, it grows steadily, reaching a maximum absolute difference of approximately 0.023851 at $t = 1.0$. This trend indicates that FVIM underestimates the solution and is less effective at tracking the growth behavior inherent in fractional systems at $\alpha = 0.7$.

Table 4: Numerical results for Problem 1 ($\alpha = 0.7$, $x = 0.3$ and $t \in [0, 1]$)

x	t	FRDTM	FVIM
0.3	0	0	0
	0.05	0.045316	0.044123
	0.1	0.090632	0.088247
	0.15	0.135948	0.132370
	0.2	0.181263	0.176493
	0.25	0.226579	0.220617
	0.3	0.271895	0.264740
	0.35	0.317211	0.308863
	0.4	0.362527	0.352987
	0.45	0.407843	0.397110
	0.5	0.453158	0.441233
	0.55	0.498474	0.485356
	0.6	0.54379	0.529480
	0.65	0.589106	0.573603
	0.7	0.634422	0.617726
	0.75	0.679738	0.661850
	0.8	0.725053	0.705973
	0.85	0.770369	0.750096
.	0.9	0.815685	0.794220
	0.95	0.861001	0.838343
	1.0	0.906317	0.882466

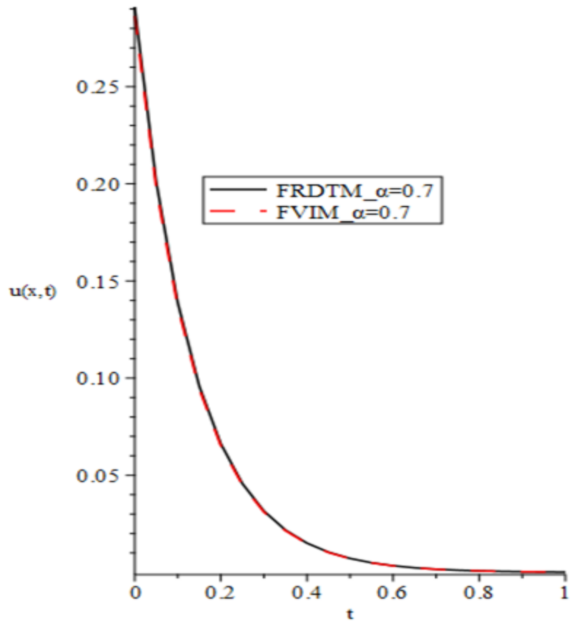

Figure 3: 2D surface plot comparing FRDTM and the FVIM approximate solutions $u(x, t)$ for $\alpha = 0.7$ at $0 < t \leq 1$ and $x = 0.3$.

Table 5: Numerical results for Problem 2 ($\alpha = 1.0$,$t \in [0,1]$ a fixed $x = \frac{\pi}{5}$ and $y = \frac{\pi}{2}$)

t	Exact Solution for $\alpha = 1$	FRDTM for $\alpha = 1$	FVIM for $\alpha = 1$	Absolute error $u(x,y,t)$ $\lvert Exac - FRDTM\rvert$	Absolute error $u(xy,,t)$ $\lvert Exact - FVIM\rvert$
0	0.7071067811866	0.7071067811866	0.7071067811866	0	0
0.1	0.5789300674675	0.5789300674675	0.5789300674675	0	0
0.2	0.4739878501174	0.4739878501179	0.4739878501171	3.00E-13	8.00E-13
0.3	0.3880684295370	0.3880684295374	0.3880684294762	6.08E-11	6.12E-11
0.4	0.3177235589369	0.3177235589370	0.3177235575109	1.43E-09	1.43E-09
0.5	0.2601300638545	0.2601300638559	0.2601300475115	1.63E-08	1.63E-08
0.6	0.2129765892745	0.2129765892743	0.2129764696973	1.20E-07	1.20E-07
0.7	0.1743710272812	0.1743710272810	0.1743703854230	6.42E-07	6.42E-07
0.8	0.1427651436882	0.1427651436889	0.1427623969720	2.75E-06	2.75E-06
0.9	0.1168938512261	0.1168938512264	0.1168839647840	9.89E-06	9.89E-06
1.0	0.0957275423705	0.0957275423704	0.0956964965107	3.10E-05	3.10E-05

Table 5 and Fig. 4 demonstrate that both approximations accurately match the exact solution at $t = 0$for fractional order $\alpha = 1$and fixed $x = \frac{\pi}{5}, y = \frac{\pi}{2}$. As time increases, a slight increase in error, ranging from 10^{-11} to 10^{-7} was observed at $t \leq 0.3$. Additionally, the errors increase more dramatically after$t = 0.7$, reaching 10^{-6} at $t = 0.9$ and 10^{-5}at $t = 1.0$. Despite this increase, both methods remain stable and closely aligned, even as the magnitude of the solution grows. The plot also shows that the FRDTM and FVIM result are most identical across all time steps.

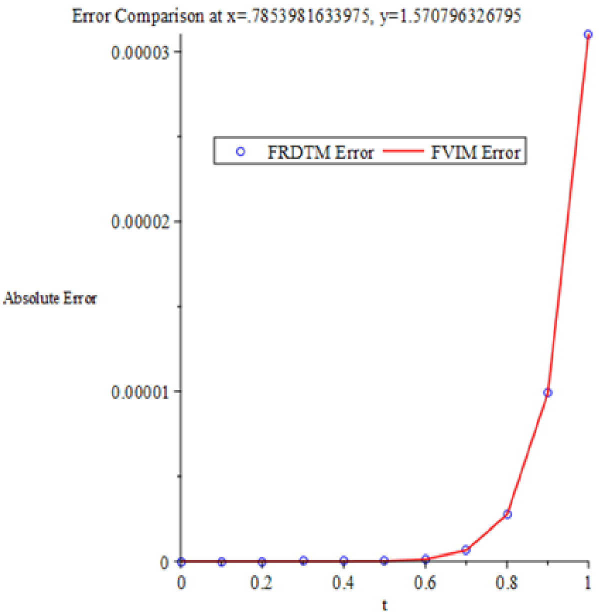

Figure 4: 2D surface plotsof the absolute errors for FRDTM and the FVIM approximate solutions $u(x, y, t)$with$\alpha = 1.0$,$t \in [0,1]$ at fixed $x = \frac{\pi}{5}$ and $y = \frac{\pi}{2}$

Table 6: Numerical results for problem 2 ($\alpha = 0.5$, $t \in [0,1]$ a fixed $x = \frac{\pi}{5}$ and $y = \frac{\pi}{2}$)

t	FRDTM	FVIM
0	0.7071067811866	0.7071067811866
0.1	0.3914714125612	0.3914714125615
0.2	0.3245576738027	0.3245576738033
0.3	0.2894948326892	0.2894948326894
0.4	0.2794438512051	0.2794438512064
0.5	0.3083688206414	0.3083688206438
0.6	0.4078368115125	0.4078368115145
0.7	0.6299293473556	0.6299293473570
0.8	1.0515197688250	1.0515197688320
0.9	1.7790529159950	1.7790529160040
1.0	2.9535799576180	2.9535799576280

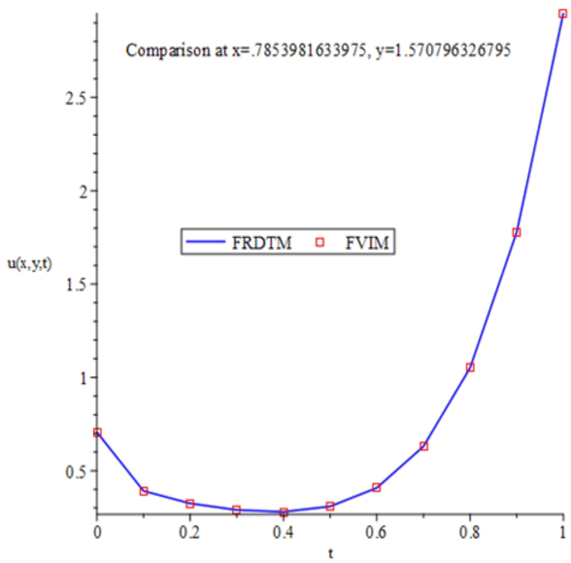

Figure 5: 2D surface plot comparing FRDTM and FVIM approximate solutions $u(x, y, t)$ for $\alpha = 0.5$, $t \in [0,1]$at fixed $x = \frac{\pi}{5}$ and $y = \frac{\pi}{2}$

The FRDTM and FVIM approximate solutions for $= 0.5$ and $t \in [0,1]$ evaluated at the fixed spatial points $x = \frac{\pi}{5}$ and $y = \frac{\pi}{2}$ are shown in Table 6, and Fig. 5 displays the corresponding plot of the findings. The solution $u(x, y, 0) \approx 0.7071067811866$ is obtained with the same values by both FRDTM and FVIM, suggesting that both approaches meet the initial condition at$t = 0$. The solution $u(x, y, t)$ first falls as time goes from $t = 0$ to $t = 1.0$, reaching a minimum 0.2794438512051 at $t = 0.4$, before starting to rise quickly from $t = 0.5$ to $t = 1.0$. As it is common with many fractional-order systems, this pattern points to a nonlinear time-dependent behavior of the solution. Furthermore, the plot demonstrates how closely the numerical values generated by the two approximations matches.

For $\alpha = 0.7$ and $t \in [0,1]$ at a fixed spatial point $x = \frac{\pi}{5}$ and $y = \frac{\pi}{2}$, it was observed in Table 7 and Fig. 6 that the FRDTM and the FVIM solution start correctly at approximately $u(x, y, 0) \approx 0.7071067811866$, which matches the exact initial condition. The differences between FRDTM and FVIM are on the order of 10^{-13} to10^{-12}, which is within machine precision, making them practically equivalent. As time increases, both approximate solutions then decrease smoothly from 0.4700471995218 at $t = 0.1$ to 0.1978999540998 at $t = 0.9$ and then start to increase rapidly at $t = 1.0$.

Table 7: Numerical results for problem 2 ($\alpha = 0.7$, $t \in [0, 1]$ a fixed $x = \frac{\pi}{5}$ and $y = \frac{\pi}{2}$)

t	*FRDTM*	*FVIM*
0	0.7071067811866	0.7071067811866
0.1	0.4700471995218	0.4700471995225
0.2	0.3751665923260	0.3751665923264
0.3	0.3146103905760	0.3146103905760
0.4	0.2716624136508	0.2716624136511
0.5	0.2395301859173	0.2395301859170
0.6	0.2150033723052	0.2150033723066
0.7	0.1968903796777	0.1968903796781
0.8	0.1857532127087	0.1857532127090
0.9	0.1842483829923	0.1842483829922
1.0	0.1978999540998	0.1978999541006

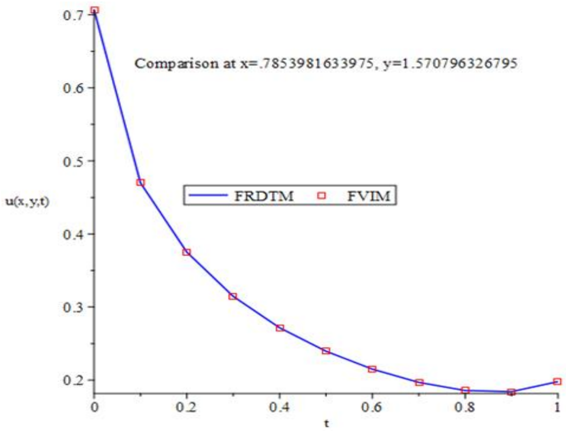

Figure 6: 2D surface plot comparing FRDTM and FVIM approximate solutions $u(x, y, t)$ for $\alpha = 0.7$, $t \in [0, 1]$ at fixed $x = \frac{\pi}{5}$ and $y = \frac{\pi}{2}$

Table 8: Numerical results for Problem 2 ($\alpha = 0.5$, $t \in [0, 1]$ a fixed $x = \frac{2\pi}{5}$ and $y = \frac{4\pi}{2}$)

t	FRDTM	FVIM
0	0.5590169943748	0.5590169943748
0.1	0.3094853256341	0.3094853256344
0.2	0.2565853703255	0.2565853703260
0.3	0.2288657605368	0.2288657605369
0.4	0.2209197619843	0.2209197619853
0.5	0.2437869581516	0.2437869581535
0.6	0.3224233095100	0.3224233095115
0.7	0.4980028756566	0.4980028756577
0.8	0.8312993685451	0.8312993685501
0.9	1.4064648231260	1.4064648231340
1.0	2.3350099793730	2.3350099793810

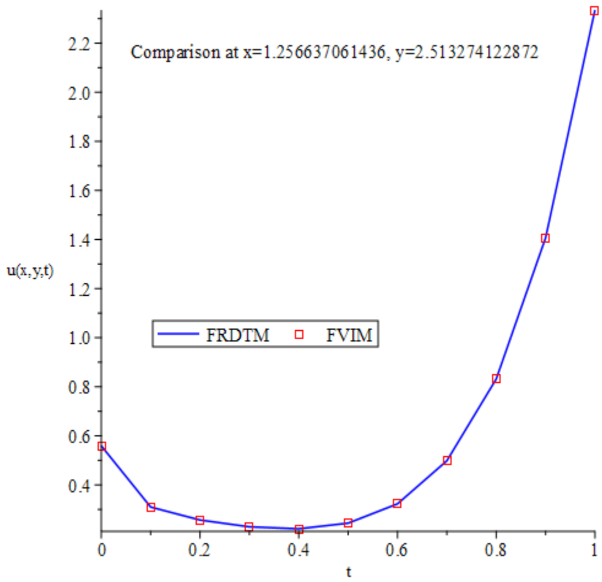

Figure 7: 2D surface plot comparing FRDTM and FVIM approximate solutions $u(x, y, t)$ for $\alpha = 0.5$, $t \in [0, 1]$ at fixed $x = \frac{2\pi}{5}$ and $y = \frac{4\pi}{5}$

As seen from Table 8 and Fig. 7 for $\alpha = 0.5$ at $t \in [0,1]$, and fixed $x = \frac{2\pi}{5}$ and $y = \frac{4\pi}{5}$, both approaches begin with the same value, $u(x, y, 0) = 0.5590169943748$, at $t = 0$, which is in consistent with the initial condition. Early on, the FRDTM and FVIM solutions exhibit a smooth decay from 0.3094853256341, 0.3094853256344 at $t = 0.1$ to 0.2437869581516, 0.2437869581535 at $t = 0.5$. At $t = 0.6$ and $t = 0.7$, both approximate solutions then rise with time. Additionally, after$t = 0.8$, the approximate solutions for both methods increase more sharply, reaching 1.40646482312600, 1.406464823134 at $t = 0.9$ and 2.3350099793730, 2.3350099793810 at $t = 1.0$.

Table 9: Numerical results for problem 2 ($\alpha = 0.7$, $t \in [0, 1]$ a fixed $x = \frac{2\pi}{5}$ and $y = \frac{4\pi}{2}$)

t	FRDTM	FVIM
0	0.5590169943748	0.5590169943748
0.1	0.3716049395680	0.3716049395686
0.2	0.2965952334384	0.2965952334387
0.3	0.2487213524438	0.2487213524438
0.4	0.2147679954488	0.2147679954490
0.5	0.1893652389655	0.1893652389652
0.6	0.1699750902754	0.1699750902764
0.7	0.1556555122892	0.1556555122895
0.8	0.1468508087133	0.1468508087135
0.9	0.1456611363646	0.1456611363646
1.0	0.1564536509495	0.1564536509501

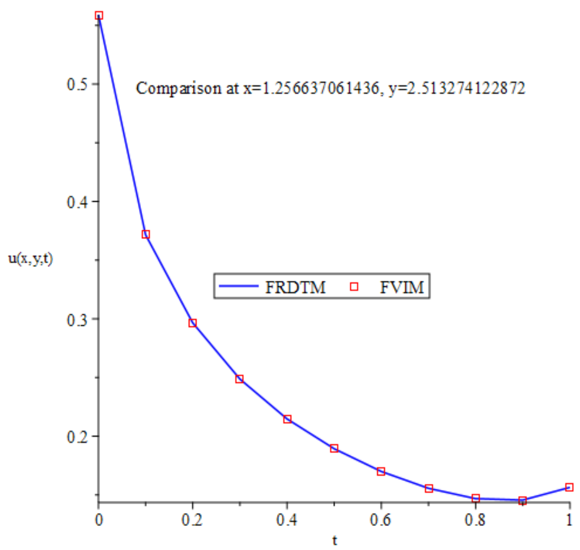

Figure 8: 2D surface plot comparing FRDTM and FVIM approximate solutions $u(x, y, t)\alpha = 0.7, t \in [0, 1]$ at fixed $x = \frac{2\pi}{5}$ and $y = \frac{4\pi}{5}$

Table 9 and Fig. 8 demonstrate that the FRDTM and FVIM solutions yielded the same value, $u(x, y, 0) = 0.5590169943748$ for $\alpha = 0.7$ at $t \in [0,1]$ and fixed $x = \frac{2\pi}{5}$ and$y = \frac{4\pi}{5}$. This indicates that both approaches meet the initial condition. From 0.3716049395680, 0.3716049395686 at $t = 0.1$ to 0.1456611363646, at $t = 0.9$, both approximations decrease steadily and smoothly before starting to rise quickly at$t = 1.0$.

Conclusion

In this work, the Fractional Reduced Differential Transform Method (FRDTM) was applied to solve fractional partial differential equations, including one- and two-dimensional fractional heat equations. To verify its reliability, the Fractional Variational Iteration Method (FVIM) was also implemented for the same problems. Numerical comparisons between FRDTM, FVIM, and the exact solutions ($\alpha = 1$) demonstrate that FRDTM achieves higher accuracy with fewer computational steps for fractional orders $\alpha = 0.2, 0.5$, and 0.7. The results further indicate that, although FVIM is useful for validation, its convergence can be slower in certain cases compared to FRDTM.

Error analysis confirms that FRDTM consistently produces smaller absolute errors than FVIM. Overall, this study establishes FRDTM as a reliable, efficient, and powerful technique for solving fractional partial differential equations, with FVIM serving as a suitable validation method. The findings support the broader application of fractional calculus in modeling complex heat phenomena and provide a foundation for future research on numerical methods for fractional PDEs.

Conflict of interest: The authors have declared that there is no conflict of interest in the manuscript.

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