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Approximate Solution of the Nonlinear Buckmaster Partial Differential Equation Using Exponential Fourth-Order Differentiable Functions

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Abstract

In this paper, the nonlinear partial differential equation, Buckmaster equation is solved using the exponential cubic B-spline collocation method (ECBSM) and the approximate solutions from this method are compared with those of the hybrid cubic B-spline collocation method (HCBSM). In order to solve the equation, linearization technique is needed to linearize the nonlinear terms of the equation. This is done by the Taylor's expansion approach. Further, the linearized equation is discretized into the fully implicit scheme and the Crank-Nicolson scheme. Three examples are used to test the proposed schemes by the fully implicit and Crank-Nicolson methods. The absolute errors of the methods are calculated and the comparison between the results of the ECBSM and the HCBSM is carried out. This is to analyze the accuracy of the methods of approximation. Both the ECBSM and HCBSM possess a free parameter which aids in determining accurate results. In general, the methods proved reliable with accuracy in approximating solutions of the equation.

Keywords: Buckmaster equation, collocation method, exponential cubic B-splines, partial differential equations, splines

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Introduction

The importance of partial differential equations (PDEs) cannot be set aside as these types of equations find applications in physical, biological and chemical phenomena. In recent times, they have found many applications in areas like image processing, economics and financial forecasting etc [1].

Buckmaster equation is a PDE of second order which is nonlinear with two nonlinear terms. It is also a parabolic PDE and basically a one-dimensional type of heat equation which has application in the field of fluid dynamics. It is used to model the thin viscous fluid sheet flow and can also be represented as a two-dimensional heat equation [2]. The general form of the Buckmaster equation is given by

$$u_t = (u^4)_{xx} + (u^3)_x \,, \quad r < x < s, 0 \le t \le T \qquad (1)$$

with boundary conditions

$$u(r,t) = \varphi_1(t), \;\; 0 \le t \le T \qquad (2a)$$

$$u(s,t) = \varphi_2(t), \quad 0 \le t \le T \qquad (2b)$$

and initial condition

$$u(x,0) = \alpha(x), \quad r < x < s. \qquad (3)$$

The Buckmaster equation has also found application in image processing in the biomedical sciences where it is in combination with the Airy equation to form the Buckmaster-Airy filter. This filter is then used for extracting vessels from digital biomedical images [3].

On the other hand, the historical development and use of spline functions to solve differential equations is traced back to the year, 1968 [4]. Bickley began the investigation on splines by applying the cubic spline to solve two-point boundary value problems involving a linear ordinary differential equation (ODE). The principle behind this method is that a spline function can be arbitrarily defined on a certain domain [5]. Successive works further improved, modified and generalized on it. For example, Fyfe [6], solved linear boundary value problems of second order using cubic splines and investigated a great deal of research questions of interest such as the effect of non-uniform spacing, deferred corrections and a net refinement procedure. Albasiny and Hoskins [7] also solved linear equations of second order using cubic splines and considered a special case with fourth-order accuracy of results to show the identity with difference schemes. Other works that were further carried out on the application of splines to solve different kinds of problems included the researches of Caglar *et al.* [8], Dehghan and Lakestani [9], Abd Hamid *et al.* [10], Abd Hamid *et al.* [11] and Khan *et al.* [12].

Amongst these splines are the exponential spline functions which have also gained unprecedented acceptance in solving differential equations. Although, these splines were first used in 1981 [13], they were neglected in the years after. McCartin [14] submitted that the exponential splines are a generalization of the semi-classical cubic spline in which the presence of certain tension parameters provides for the adjustment of the tautness of individual spline segments and that the lack of a suitable scheme of tension parameter was the reason for the lack of application of the functions. This hitch was removed by the establishment of a well-founded algorithm thereby giving room to the widespread application of the exponential splines in a variety of data fitting and geometric problems in computational fluid dynamics [15].

As a result, Ersoy and Dag [16–18] solved the Korteweg-de Vries equation, Reaction-Diffusion equation and Kuramoto-Sivashinsky equation using the ECBSM, Zhu *et al.* [19] presented a collocation method for the fractional sub-diffusion equation based on the exponential B-spline basis function and Singh *et al.* [20] proposed an ECBSM for the numerical solution of second-order. Most recently, Shukla &Tamsir [21] proposed an algorithm for solving the multi-dimensional convection-diffusion equations based on the exponential cubic B-spline. This algorithm is a quadrature method with a modification and is used to approximate the 2D and 3D convection-diffusion equation. The method yielded better results than the results of most methods in literature. Additionally, the method is economically viable in terms of computational cost and it is simple to implement.

Just in the recent years, the Buckmaster equation had been given attention and as a consequence, a few works have been done on it. As a nonlinear PDE, it has been solved by some analytical and numerical methods such as the Finite Volume Method (FVM) [22], the q-Homotopy Analysis Method (q-HAM) [23], Cubic B-spline Interpolation Method (CBSM) and Cubic Trigonometric B-spline Interpolation Method (CTBSM) [24] etc. The numerical solutions of the Buckmaster equation proffered using the collocation method based on the cubic B-splines and trigonometric cubic B-spline functions [24] caught our interest. The results of the method for some problems considered were compared with analytical solutions and those of FDM. The study concluded that the schemes are capable and generate results that are of better accuracy than the FDM.

These findings have encouraged us to choose no other numerical method to make comparison with but the HCBSM.

Materials and Methods

Splines

Splines are commonly considered as piecewise functions. These functions, most especially, the B-splines have found applications in geometric representation, computer-aided design, computer graphics and several other fields [25]. In this work, the Exponential cubic B-spline is considered and used in the collocation method to solve the Buckmaster Equation as an interpolating function for the space derivatives.

Exponential cubic B-spline function

The exponential spline is defined in similitude with the theory of cubic splines to a beam in which a tension parameter p is added to the differential equation governing the system [13]. Given an interval $[r,s]$ that is assumed to be uniformly divided into N subintervals $[x_j, x_{j+1}], (j = 0,1,2, \dots, N-1)$, i.e., $r = x_0 < x_1 < x_2 < \cdots < x_N = s$. The exponential cubic B-spline, $\hat{B}_{3,j}(x)$ with knots at the grid points $x_j = r + jh$, where $j = 0,1,2, \dots, N)$ and $h = (s-r)/N$ together with fictitious knots $x_{-3}, x_{-2}, x_{-1}, x_{N+1}, x_{N+2}, x_{N+3}$ outside the problem domain $[r,s]$ can be defined as [18, 21];

$$\hat{B}_{3,j}(x) = \begin{cases} b_2\left(\left(x_{j-2}-x\right)-\frac{1}{p}\left(\sinh\left(p\left(x_{j-2}-x\right)\right)\right)\right), & [x_{j-2}, x_{j-1}) \\ a_1 + b_1(x_j - x) + c_1 \exp\left(p(x_j - x)\right) + d_1 \exp\left(-p(x_j - x)\right), & [x_{j-1}, x_j) \\ a_1 + b_1(x - x_j) + c_1 \exp\left(p(x - x_j)\right) + d_1 \exp\left(-p(x - x_j)\right), & [x_j, x_{j+1}) \\ b_2\left(\left(x - x_{j+2}\right)-\frac{1}{p}\left(\sinh\left(p\left(x - x_{j+2}\right)\right)\right)\right), & [x_{j+1}, x_{j+2}) \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

where, $a_1 = \frac{phc}{phc-s}$, $b_1 = \frac{p}{2}\left[\frac{c(c-1)+s^2}{(phc-s)(1-c)}\right]$, $b_2 = \frac{p}{2(phc-s)}$,

$$c_1 = \frac{1}{4}\left[\frac{\exp(-ph)(1-c) + s(\exp(-ph)-1)}{(phc-s)(1-c)}\right]$$

$$d_1 = \frac{1}{4}\left[\frac{\exp(ph)(c-1) + s(\exp(ph)-1)}{(phc-s)(1-c)}\right]$$

and $c = \cosh(ph)$, $s = \sinh(ph)$, p is a free parameter.

Linearization

The linearization process entails converting the nonlinear terms of the Buckmaster equation to linear terms. This process is achieved by the use of Taylor's expansion. This process is crucial in the implementation of the implicit schemes, namely, the Crank-Nicolson scheme and fully implicit scheme adopted in this work.

The nonlinear terms of the Buckmaster Equation are$(u^4)_{xx}$ and $(u^3)_x$, and their respective linearization are given below as;

$$((u^4)_{xx})_j^{k+1} \approx 4((u^3)_{xx})_j^k (u_{xx})_j^{k+1} - 3((u^4)_{xx})_j^k \quad (5)$$

$$((u^3)_x)_j^{k+1} \approx 3((u^2)_x)_j^k (u_x)_j^{k+1} - 2((u^3)_x)_j^k \quad (6)$$

Temporal discretization

The discretization carried out in this work is called the temporal discretization as it is done by the use of forward difference approximation on the time derivatives only. We have the θ −weighted scheme for the Buckmaster equation given as below [24];

$$(u_t)_j^k = \theta\mathfrak{P}_j^{k+1} + (1-\theta)\mathfrak{P}_j^k \quad (7)$$

where, $\mathfrak{P}_j^k = ((u^4)_{xx})_j^k + ((u^3)_x)_j^k + (h(x,t))_j^k$

Substituting Equations (5) and (6) as well as the forward difference approximation for the time difference term into Equation (7) and simplifying further, we have;

$$u_j^{k+1} - \theta\Delta t(4((u^3)_{xx})_j^k(u_{xx})_j^{k+1} + 3((u^2)_x)_j^k(u_x)_j^{k+1} + h(x,t)_j^{k+1}) = u_j^k + \theta\Delta t(-3((u^4)_{xx})_j^k - 2((u^3)_x)_j^k) + (1-\theta)\Delta t(((u^4)_{xx})_j^k + ((u^3)_x)_j^k + h(x,t)_j^k) \quad (8)$$

Lastly, by the use of chain rule, Equation (8) is expanded to give;

$$u_j^{k+1} - \theta\Delta t((24u(u_x)^2 + 12u^2u_{xx})_j^k(u_{xx})_j^{k+1}) - \theta\Delta t((6uu_x)_j^k(u_x)_j^{k+1}) - \theta\Delta th(x,t)_j^{k+1} = u_j^k - \theta\Delta t((36u^2(u_x)^2 + 12u^3u_{xx})_j^k) - \theta\Delta t((6u^2u_x)_j^k) + (1-\theta)\Delta t((12u^2(u_x)^2 + 4u^3u_{xx})_j^k) + (1-\theta)\Delta t((3u^2u_x)_j^k) + (1-\theta)\Delta th(x,t)_j^k \quad (9)$$

In this work, we consider the fully implicit scheme where $\theta = 1$ and the Crank-Nicolson scheme where $\theta = 0.5$.

Exponential B-spline collocation method

In developing the collocation method using the exponential cubic B-spline basis function, an approximate solution to the analytical solution is sought as;

$$u_N(x,t) = \sum_{j=-1}^{N+1} \sigma_j(t)\hat{B}_{3,j}(x) \quad (10)$$

Where $\sigma_j(t)$ are the time-dependent unknowns to be determined and the $\hat{B}_{3,j}(x)$ are cubic exponential B-spline basis functions of degree 3, order 4 defined in (4).

The approximate solution (10) and its derivatives with respect to x at the knot points (x_j, t_n) are given as follows;

$$(u)_j^k = u(x_j,t_k) = \lambda_1\sigma_{j-1}^k + \sigma_j^k + \lambda_1\sigma_{j+1}^k \quad (11)$$

$$(u_x)_j^k = u'(x_j,t_k) = \lambda_2\sigma_{j-1}^k + \lambda_3\sigma_{j+1}^k \quad (12)$$

$$(u_{xx})_j^k = u''(x_j,t_k) = \lambda_4\sigma_{j-1}^k + \lambda_5\sigma_j^k + \lambda_4\sigma_{j+1}^k \quad (13)$$

where, $\lambda_1 = \frac{s-ph}{2(phc-s)}$, $\lambda_2 = \frac{p(1-c)}{2(phc-s)}$, $\lambda_3 = \frac{p(c-1)}{2(phc-s)}$, $\lambda_4 = \frac{p^2s}{phc-s}$, $\lambda_5 = -2\lambda_4$

and $c = \cosh(ph)$, $s = \sinh(ph)$, p is a free parameter.

Now, we consider (9) and simplify it further letting $\mu = u_j^k$, $\rho = (u_x)_j^k$ and $\tau = (u_{xx})_j^k$ to have;

$$u_j^{k+1} + (u_x)_j^{k+1}[-\theta B_1] + (u_{xx})_j^{k+1}[-\theta B_2] - \theta\Delta th(x,t)_j^{k+1} = u_j^k + (u_x)_j^k[-\theta B_3 + (1-\theta)\Delta tB_4] + (u_{xx})_j^k[-\theta B_5 + (1-\theta)B_6] + (1-\theta)\Delta th(x,t)_j^k \quad (14)$$

Where: $B_1 = 6\Delta t\mu\rho$, $B_2 = \Delta t(24\mu\rho^2 + 12\mu^2\tau)$, $B_3 = \Delta t(36\mu^2\rho + 6\mu^2)$, $B_4 = \Delta t(12\mu^2\rho + 3\mu^2)$, $B_5 = 12\Delta t\mu^3$, $B_6 = 4\Delta t\mu^3$

Equations (11), (12) and (13) are then substituted into (14) which yields a$(n+1)$ linear system with$(n+3)$unknowns and the system is condensed and written in the matrix form;

$$PC^{n+1} = QC^n + H \quad (15)$$

In order to obtain a unique solution, the boundary conditions (2a) and (2b) are approximated as;

$$u_0^{k+1} = \lambda_1\sigma_{-1}^{k+1} + \sigma_0^{k+1} + \lambda_1\sigma_1^{k+1} = \varphi_1(t_{k+1}) \quad (16a)$$

$$u_N^{k+1} = \lambda_1\sigma_{N-1}^{k+1} + \sigma_N^{k+1} + \lambda_1\sigma_{N+1}^{k+1} = \varphi_2(t_{k+1}) \quad (16b)$$

By the use of Equation (9) with the Equations (16a) and (16b), a solvable system of $(n+3)$ linear equations involving$(n+3)$ unknowns is obtained and transformed into a tridiagonal matrix system with dimension $(n+3) \times (n+3)$ which is solved by the method of Thomas Algorithm with the aid of Mathematica software.

Initial state

The initial vector σ^0 which is needed in order to obtain the solution parameters is gotten by the use of the initial condition (3) and boundary values of the derivatives of the initial condition as below;

$$u(x_j,0) = \alpha(x_j), \quad j = 0,1,2,\dots,N, \quad (17a)$$

$$u_x(x_0,0) = \alpha'(x_0), \quad \text{for } j = 0 \quad (17b)$$

and;

$$u_x(x_N,0) = \alpha'(x_N), \quad \text{for } j = N \quad (17c)$$

Hence, a tridiagonal matrix system of $(n+3) \times (n+3)$ dimension is formed from the above operation as thus;

$$E\alpha^0 = G \quad (18)$$

The matrix α^0 is generated by solving (18).

Results and Discussion

This section reveals the numerical solutions of the nonlinear Buckmaster equation by the ECBSM in comparison with the results generated by the HCBSM. To test the efficiency and accuracy of the ECBSM, three problems involving the Buckmaster equation have been examined. The above two methods considered in this work are tested on these problems using the fully implicit scheme with θ −value as 1 and the Crank-Nicolson scheme with θ −value as 0.5.

Problem 1

Consider the following nonlinear parabolic IBVP,

$$u_t(x,t) - (u^4)_{xx}(x,t) - (u^3)_x(x,t) = -12x^2\cos^4 t - 3x^2\cos^3 t - x\sin t, \quad x \in (0,1), t \in \mathbb{R}^+$$

subject to the boundary conditions,

$$u(0,t) = 0, \quad u(1,t) = \cos t, \quad t \in \mathbb{R}^+$$

and initial condition,

$$u(x,0) = x, \quad x \in (0,1)$$

The exact solution to this problem is $u(x,t) = x\cos t$. For the spatial step size, time interval and highest time level, we used $h = 0.2$, $\Delta t = 0.01$ and $T = 0.05$ respectively. Brute-force method is applied on the results of problem 1 using ECBSM with $\theta = 1$ (i.e. fully implicit scheme) and $\theta = 0.5$ (i.e. Crank-Nicolson scheme) from $p = -10$ to $p = 10$ with p −step size 0.0001. The approximate solutions acquired by using HCBSM with the aid of Mathematica software and the approximate solutions by ECBSM for fully implicit scheme ($\theta = 1$) and Crank-Nicolson scheme ($\theta = 0.5$) are tabulated in Table 1 alongside the values of the exact solution.

From Table 1, the results by the two numerical methods show that the approximate solutions by ECBSM for both fully implicit scheme and Crank-Nicolson scheme with $p = 8.3732$ and $p = 3.0977$ respectively are closer to the exact solutions compared to the ECBSM except for the solution at $x = 0.2$.

Further comparison between HCBSM and ECBSM is presented in terms of the absolute error, Euclidean error (L_2 −norm) and the maximum absolute error (L_∞ −norm) for both the fully implicit scheme and the Crank-Nicolson scheme as presented in Table 2. Fig. 1 shows the plot of error for fully implicit scheme while Fig. 2 shows the plot of error for the Crank-Nicolson scheme for Problem 1.

Table 1: Numerical results for Problem 1

x	Fully Implicit Scheme ($\theta = 1$)		Crank-Nicolson Scheme ($\theta = 0.5$)		Exact Solution
	HCBSM $\gamma = 64.709$	**ECBSM** $p = 8.3732$	**HCBSM** $\gamma = 64.2167$	**ECBSM** $p = 3.0977$	
0.0	0.000000	0.000000	0.000000	0.000000	0.00000
0.2	0.196306	0.196300	0.196902	0.199872	0.19975
0.4	0.400332	0.396459	0.401231	0.399582	0.39950
0.6	0.602650	0.601042	0.602759	0.599128	0.59925
0.8	0.802445	0.797028	0.802249	0.798880	0.79900
1.0	0.998750	0.998750	0.998750	0.998750	0.99875

Table 2: Absolute, L_∞ and L_2 errors for Problem 1 with $T = 0.05$

x	Fully Implicit Scheme ($\theta = 1$) Absolute Error		Crank-Nicolson Scheme ($\theta = 0.5$) Absolute Error	
	HCBSM $\gamma = 64.709$	**ECBSM** $p = 8.3732$	**HCBSM** $\gamma = 64.2167$	**ECBSM** $p = 3.0977$
0.0	0.000000	0.000000	0.000000	0.000000
0.2	3.4441E-03	3.4498E-03	2.8484E-03	1.2203E-04
0.4	8.3182E-03	1.0409E-03	1.7314E-03	8.1821E-05
0.6	3.4002E-03	1.7917E-03	3.5086E-03	1.2194E-04
0.8	3.4447E-03	1.9724E-03	3.2485E-03	1.2005E-04
1.0	0.000000	0.000000	0.000000	0.000000
L_∞	3.4447E-03	3.4498E-03	3.5086E-03	1.2203E-04
L_2	5.9984E-03	4.4816E-03	5.8288E-03	2.2554E-04

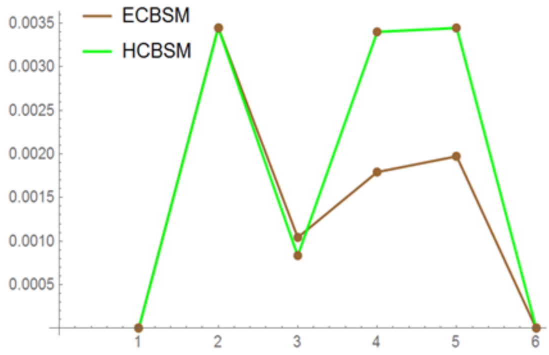

Figure 1: Plot of error for fully implicit scheme

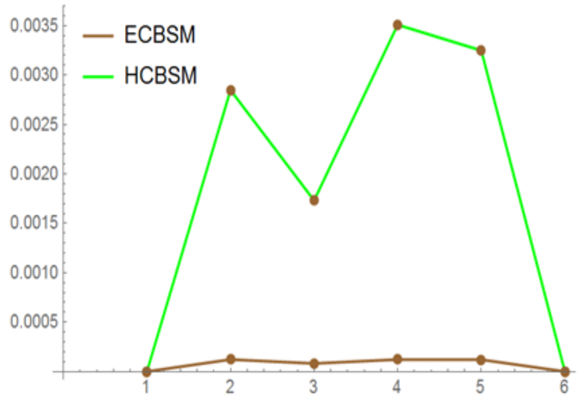

Figure 2: Plot of error for Crank-Nicolson scheme

Problem 2

Consider the following parabolic IBVP,

$$u_t(x,t) - (u^4)_{xx}(x,t) - (u^3)_x(x,t) = -12x^2e^{4t} - 3x^2e^{3t} + xe^t, \quad x \in (0,1), t \in \mathbb{R}^+$$

subject to the boundary conditions,

$$u(0,t) = 0, \quad u(1,t) = e^t, \quad t \in \mathbb{R}^+$$

and initial condition,

$$u(x,0) = x, \quad x \in (0,1)$$

The exact solution to this problem is $u(x,t) = xe^t$. For the spatial step size, time interval and highest time level, the values $h = 0.2$, $\Delta t = 0.01$ and $T = 0.05$ respectively are maintained.In this problem too, the brute-force method is applied on the results using ECBSM with $\theta = 1$ (i.e. fully implicit scheme) and $\theta = 0.5$ (i.e. Crank-Nicolson scheme) from $p = -10$ to $p = 10$ with p −step size 0.0001 and the best values of the free parameter, p are gotten to be $p = -9.7478$ and $p = -8.3819$ respectively for the schemes. The approximate solutions acquired by using HCBSM and ECBSM for fully implicit scheme ($\theta = 1$) and Crank-Nicolson scheme ($\theta = 0.5$) are tabulated in Table 3 alongside the values of the exact solution.

From Table 3, the results by the two numerical methods show that the approximate solutions by HCBSM for both fully implicit scheme and Crank-Nicolson scheme with $\gamma = 58.6282$ and $\gamma = 58.4378$ respectively are closer to the exact solutions compared to the ECBSM.
Comparison between HCBSM and ECBSM is presented in terms of the L_2 −norm and L_∞ −norm for both the fully implicit scheme and the Crank-Nicolson scheme as presented in Table 4. Fig. 3 shows the plot of error for fully implicit scheme while Fig. 4 shows the plot of error for the Crank-Nicolson scheme for Problem 2.

Table 3: Numerical Results for Problem 2

x	Fully Implicit Scheme (θ = 1)		Crank-Nicolson Scheme (θ = 0.5)		Exact Solution
	HCBSM $\gamma = 58.6282$	ECBSM $p = -9.7478$	HCBSM $\gamma = 58.4378$	ECBSM $p = -8.3819$	
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.2	0.216099	0.217033	0.215943	0.207215	0.210254
0.4	0.419057	0.414838	0.419003	0.414599	0.420508
0.6	0.624918	0.617475	0.625074	0.621953	0.630763
0.8	0.835296	0.828419	0.835586	0.833772	0.841017
1.0	1.051270	1.051270	1.051270	1.051270	1.051270

Table 4: Absolute, L_∞ and L_2 errors for Problem 2 with $T = 0.05$

x	Fully Implicit Scheme (θ = 1) Absolute Error		Crank-Nicolson Scheme (θ = 0.5) Absolute Error	
	HCBSM $\gamma = 58.6282$	ECBSM $p = -9.7478$	HCBSM $\gamma = 58.4378$	ECBSM $p = -8.3819$
0.0	0.000000	0.000000	0.000000	0.000000
0.2	5.8447E-03	6.7793E-03	5.6887E-03	3.0389E-03
0.4	1.4516E-03	5.6709E-03	1.5051E-03	5.9097E-03
0.6	5.8447E-03	1.3288E-02	5.6884E-03	8.8092E-03
0.8	5.7212E-03	1.2598E-02	5.4310E-03	7.2453E-03
1.0	0.000000	0.000000	0.000000	0.000000
L_∞	5.8447E-03	1.3288E-02	5.6887E-03	8.8092E-03
L_2	1.0157E-02	2.0332E-02	9.8224E-03	1.3201E-02

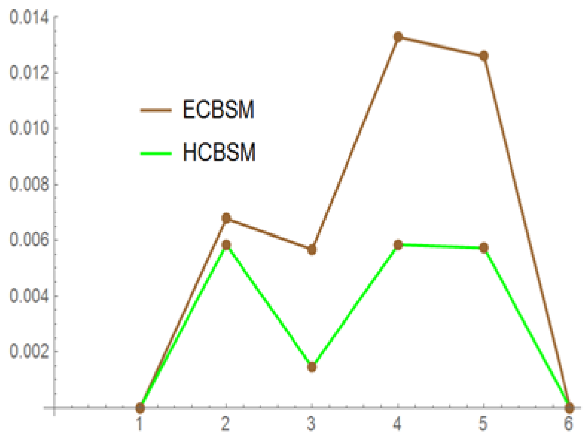

Figure 3: Plot of error for fully implicit scheme

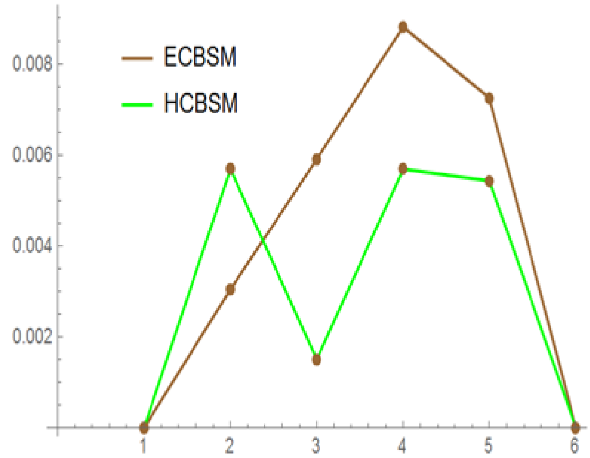

Figure 4: Plot of error for Crank-Nicolson scheme

Problem 3
Consider the following problem of non-homogenous Buckmaster equation, $u_t(x,t) - (u^4)_{xx}(x,t) - (u^3)_x(x,t) = -12x^2(1+t)^4 - 3x^2(1+t)^3 + x,\ \ x \in (0,1), t \in \mathbb{R}^+$
subject to the boundary conditions,

$$u(0,t) = 0, \quad u(1,t) = 1 + t, \quad t \in \mathbb{R}^+$$

and initial condition,

$$u(x,0) = x, \quad x \in (0,1)$$

The exact solution to this problem is $u(x,t) = x(1+t)$.The brute-force method is applied on the results of problem 3 using ECBSM with $\theta = 1$ (i.e. fully implicit scheme) and $\theta = 0.5$ (i.e. Crank-Nicolson scheme) from $p = -10$ to $p = 10$ with p −step size 0.0001 and the best values of the free parameter, p are obtained as$p = \pm 8.5296$ and $p = -8.3825$ respectively for the schemes. For the spatial step size, time interval and highest time level, we used $h = 0.2$, $\Delta t = 0.01$ and $T = 0.05$ respectively. The approximate solutions of HCBSM and ECBSM for fully implicit scheme $(\theta = 1)$ and Crank-Nicolson scheme $(\theta = 0.5)$ are tabulated in Table 5.

Table 4: Numerical results for Problem 3

x	Fully Implicit Scheme (θ = 1)		Crank-Nicolson Scheme (θ = 0.5)		Exact Solution
	HCBSM $\gamma = 64.6753$	ECBSM $p = \pm 8.5296$	HCBSM $\gamma = 64.2172$	ECBSM $p = -8.3825$	
0.0	0.000000	0.000000	0.000000	0.000000	0.000000
0.2	0.206430	0.209028	0.206732	0.210597	0.210000
0.4	0.420854	0.415619	0.421625	0.419875	0.420000
0.6	0.633519	0.621413	0.633572	0.626113	0.630000
0.8	0.843571	0.828915	0.843640	0.831882	0.840000
1.0	1.050000	1.050000	1.050000	1.050000	1.050000

Table 5: Absolute, L_∞ and L_2 errors for Problem 3 with $T = 0.05$

x	Fully Implicit Scheme (θ = 1) Absolute Error		Crank-Nicolson Scheme (θ = 0.5) Absolute Error	
	HCBSM $\gamma = 64.6753$	ECBSM $p = \pm 8.5296$	HCBSM $\gamma = 64.2172$	ECBSM $p = -8.3825$
0.0	0.000000	0.000000	0.000000	0.000000
0.2	3.5705E-03	9.7186E-03	3.2681E-03	5.9656E-04
0.4	8.5413E-04	4.3814E-03	1.6247E-03	1.2499E-04
0.6	3.5191E-03	8.5874E-03	3.5722E-03	3.8870E-03
0.8	3.5709E-03	1.1085E-02	3.3639E-03	8.1178E-03
1.0	0.000000	0.000000	0.000000	0.000000
L_∞	3.5709E-03	1.1085E-02	3.5722E-03	8.1178E-03
L_2	6.2140E-03	1.4723E-02	6.1152E-03	9.0210E-03

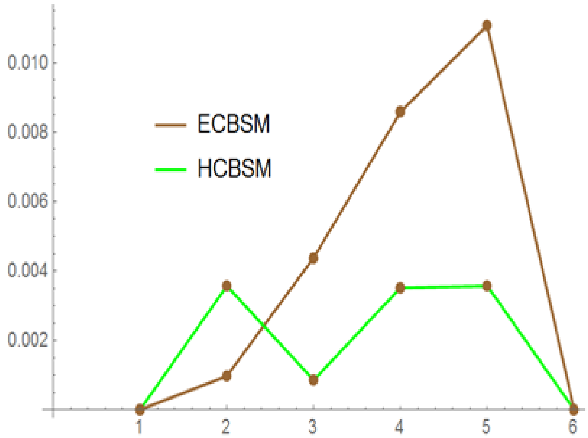

Figure 5: Plot of error for fully implicit scheme

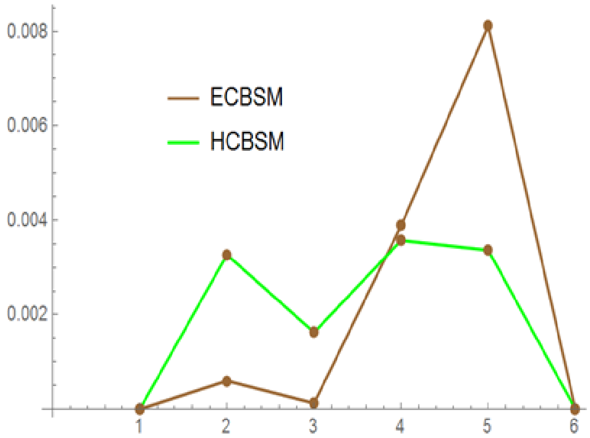

Figure 6: Plot of error for Crank-Nicolson scheme

From Table 5, the results by the two numerical methods show that the approximate solutions by HCBSM for both fully implicit scheme and Crank-Nicolson scheme except for the point $x = 0.4$ with $\theta = 0.5$ are closer to the exact solutions compared to the ECBSM.

In this example too, we comparethe results between HCBSM and ECBSM in terms of the absolute error (L_2 −norm) and the maximum absolute error (L_∞ −norm) for both the fully implicit scheme and the Crank-Nicolson scheme as presented in Table 6. Fig. 5 shows the plot of error for fully implicit scheme while Fig. 6 shows the plot of error for the Crank-Nicolson scheme for Problem 3.

Conclusion

The Buckmaster equation just like many other partial differential equations is a model of some processes which can better be understood and explained if the equation is solved. These solutions can be arrived at by some analytical methods but in most situations, these analytical solutions are difficult to come by. Hence, the need for numerical methods that can be used as alternatives to the analytical ones. This work discusses the numerical solutions of the nonlinear non-homogenous Buckmaster equation using the collocation method of exponential cubic B-spline. With the aid of the Mathematica software, this equation is solved by the ECBSM and the solutions obtained are compared with the solutions obtained by the HCBSM. It is observed that both methods are reliable and effective in solving the Buckmaster equation.

Conflict of interest: The Authors have declared that there is no conflict of interest in this study.

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