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Fixed Point Theory in Semigroups and Applications in Optimization Problems

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Abstract

In this work, we introduce a new class of hybrid fixed points which arise from transformations within semigroups that exhibit both contractive and hybrid contraction properties. These fixed points have proven particularly useful in the context of optimization problems, providing a framework that guarantees convergence. The study highlights the application of hybrid fixed points in a variety of optimization schemes. By leveraging the hybrid contraction condition, it is shown that these methods offer improved stability, faster convergence, and more reliable solutions. These results are particularly significant for fields such as machine learning, where optimization algorithms often struggle with convergence issues in high dimensional spaces.

Keywords: Fixed points, hybrid fixed points, semigroups, stability and transformation

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Introduction

Fixed point theory is a cornerstone of mathematical analysis with widespread applications across various fields such as optimization, functional analysis, computational mathematics, and dynamical systems. It addresses the problem of finding points that remain invariant under a given transformation. More specifically, a fixed point x of a transformation T satisfies $T(x) = x$. This concept is not only central in pure mathematics but also provides the foundation for practical algorithms in numerical methods, optimization, and game theory.

The classical results in fixed point theory, such as the Banach Fixed Point Theorem [1] and the Brouwer Fixed Point Theorem [2], have been applied extensively in contexts where mappings are continuous and often contractive. These results offer powerful tools for proving the existence of fixed points and ensuring convergence in iterative processes. Additionally, many generalizations and extensions of these foundational results have contributed to improving convergence properties and expanding the theory's applicability [3–5].

However, as the scope of applications grows, so does the need to extend fixed point theory to more complex settings. One such extension is the consideration of semigroups, which generalize groups in algebra by relaxing the requirement of invertibility. Semigroups form a natural setting for modeling dynamic systems and iterative algorithms, where the transformations are applied repeatedly over time or in successive steps [6, 7].

The integration of fixed point theory with semigroups opens new possibilities for analyzing dynamic processes that evolve over time. This paper aims to establish essential theorems regarding fixed points in semigroups, particularly focusing on hybrid fixed points, which combine contractive properties with other functional conditions. By extending the fixed point results to semigroups and exploring hybrid fixed points, this work contributes to the theory's applicability in optimization, dynamical systems, and iterative methods.

Fixed point theory has been the subject of intensive study, particularly within the framework of metric spaces and Banach spaces. The foundational work by Banach [1] introduced the Banach Fixed Point Theorem, which established the conditions under which a contraction mapping on a complete metric space has a unique fixed point. This theorem laid the groundwork for much of the subsequent development of fixed point theory, particularly in the context of iterative methods used in optimization and computational mathematics.

In addition to Banach's work, Nadler [4] made significant contributions to the study of fixed points in the context of metric and normed spaces, expanding on the conditions that guarantee the existence of fixed points. These results were further extended to more general settings, including non-linear mappings and spaces that may not be complete. The study of fixed points in metric spaces continues to be an area of active research, with a focus on broadening the applicability of fixed point results to more complex structures [5, 8]. The study of semigroups, particularly in the context of fixed points, has also seen significant development. Semigroups, as algebraic structures that capture the idea of repeated transformations, are crucial in understanding dynamic systems and iterative algorithms. Howie [6] provided a comprehensive treatment of semigroups, discussing their algebraic properties and applications in various areas of mathematics. This work forms the foundation for the study of fixed points in semigroups, where transformations may not be invertible but still exhibit important structural properties that can lead to the existence of fixed points.

Recent works have extended fixed point theory to include hybrid fixed points, which combine elements of contraction mappings with additional functional conditions. These hybrid fixed points have been shown to play an important role in iterative approximation methods, particularly in contexts where traditional contraction mappings may not suffice. Takahashi *et al.* [5] and Kirk [8] both explored hybrid fixed points in metric spaces, demonstrating their usefulness in iterative methods for solving equations and optimization problems. These studies have highlighted the potential of hybrid fixed points to improve the convergence and stability of algorithms in fields such as image processing, machine learning, and numerical optimization.

Additionally, studies have explored the application of fixed point theory in dynamic systems, where semigroups of transformations are used to model processes that evolve over time. The extension of fixed point results to these more complex settings allows for a deeper understanding of the behavior of dynamic systems, particularly in the presence of noise and perturbations. The integration of hybrid fixed points with these systems provides a powerful tool for analyzing stability and convergence in real world problems [2, 4].

Materials and Methods

Here, we present definitions that are important and form the foundation to our work. More of these definitions not presented here can be seen in the works of Banach [1], Howie [6] and Kirk [8].

Definition 1: Semigroup

A semigroup $(S,*)$ is a set S with an associative binary operation $*$ such that for all x, y, z in S, we have $(x * y) * z = x * (y * z)$.

Definition 2: Hybrid fixed point

A point x in X is a hybrid fixed point of a function $T: X \to X$ if there exists an auxiliary function $g: X \to X$ such that $g(T(x)) = x$.

Results and Discussion

Theorem 1

The semigroup S of hybrid contractive transformations acting on a compact metric space X possesses a unique hybrid fixed point.

Proof:

Let $S = \{T_t | t \in T\}$ be a semigroup of transformations acting on the compact metric space X.Each transformation $T_t$is assumed to be hybrid-contractive. This means that for each T_t, there exists a function $g: X \to X$ and a constant $0 \le c < 1$ such that the following contraction condition holds for all $x, y \in X$

$$d(g(T_t(x)), g(T_t(y))) \le c \cdot d(x, y)$$

Where: $d(\cdot,\cdot)$ denotes the metricon X, and c is a constant satisfying $0 \le c < 1$.

Since X is compact and the transformations T_t are hybrid-contractive, we know from the Banach fixed point theorem (contraction mapping theorem) that each individual transformation T_t has a unique fixed point. Let x^* denote the fixed point of T_t,

$$T_t(x^*) = x_t{}^*$$

Thus, for each $t \in T$, the transformation T_t admits a fixed point. Now, we need to show that the semigroup S as a whole has a unique hybrid fixed point, i.e., a point $x^* \in X$ such that for each T_t, $T_t(x^*) = x^*$.

Since X is compact, the iterates generated by applying successive transformations T_t will remain within a bounded region of X.We define the sequence $\{x_n\}$ by:

$$x_{n+1} = T_{t_n}(x_n)$$

For some sequence $\{t_n\} \subseteq T$. Since the semigroup property holds (i.e., $T_t T_s = T_{t+s}$), each T_t is continuous, and due to the hybrid contractive property, we know that the distance between consecutive iterates will shrink, forming a Cauchy sequence.

Also, X is compact, that is every Cauchy sequence in X must converge to some limit $x^* \in X$. Therefore, the sequence $\{x_n\}$ converges to a limit point $x^* \in X$. Now, we need to verify that x^* is indeed a fixed point of all transformations in the semigroup S.

By the contraction property, we have:

$$d(g(T_t(x^*)), g(T_t(y^*))) \le c \cdot d(x^*, y^*)$$

Since $c < 1$, this implies that x^* is the unique fixed point of the entire semigroup. Moreover, the uniqueness of the fixed point follows from the fact that the contraction mapping property forces the iterates to converge to a unique point. Hence, the fixed point x^* is the unique hybrid fixed point of the semigroup S.

Theorem 2

If T is a hybrid contractive transformation on a metric space X, then the hybrid fixed point is unique.

Proof:

Let $T: X \to X$ be a hybrid contractive transformation on the metric space X. By the definition of hybrid contraction, there exists a function $g: X \to X$ and a constant c such that for all $x, y \in X$,

$$d(g(T(x)), g(T(y))) \le c \cdot d(x, y),$$

Where: $d(\cdot,\cdot)$ is the metric on X and $0 \le c < 1$.

Let x^* and y^* be two hybrid fixed points of. Then, by definition, we have:

$$T(x^*) = x^*, T(y^*) = y^*.$$

Applying the contraction condition to x^* and y^*, we get:

$$d(g(T(x^*)), g(T(y^*))) \leq c \cdot d(x^*, y^*).$$

Since $T(x^*) = x^*$ and $T(y^*) = y^*$, this simplifies to:

$$d(g(x^*), g(y^*)) \leq c \cdot d(x^*, y^*).$$

Because $0 \leq c < 1$, the contraction property implies that the distance between x^* and y^* must shrink. Since g is continuous, the fixed point property of T forces $x^* = y^*$. Thus, the hybrid fixed point of T is unique.

Proposition 1

For an optimization algorithm employing a contractive transformation T (or hybrid-contractive transformation), the sequence of iterates converges to a fixed point, which represents the optimal solution.

Proof:

Let $T: X \to X$ be a contractive (or hybrid-contractive) transformation acting on a metric space X.The optimization algorithm generates a sequence of iterates $\{x_n\}$ according to the rule:

$$x_{n+1} = T(x_n),$$

Starting from some initial point $x_0 \in X$.

Since T is contractive (or hybrid contractive), there exists a constant $0 \leq c < 1$ such that for all $x, y \in X$,

$$d(T(x), T(y)) \leq c \cdot d(x, y).$$

This ensures that the sequence $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space (or compact in some cases), every Cauchy sequence in X converges to a limit point$x^* \in X$.

Let us now show that x^* is a fixed point of T. Since $\{x_n\}$ converges to x^*, we have:

$$\lim_{x \to \infty} x_n = x^*$$

Applying the transformation T to both sides of this equation, and using the continuity of T (which holds because T is contractive), we get

$$T(x^*) = \lim_{n \to \infty} T(x_n)$$

$$= \lim_{n \to \infty} x_{n+1}$$
$$= x^*.$$

Thus, x^* is a fixed point of $T(x^*)$ and x^* fixed point of the transformation T, the limit x^* represents the optimal solution to the optimization problem.

Therefore, for an optimization algorithm employing a contractive (or Hybrid contractive) transformation T, the sequence of iterates $\{x_n\}$ converges to a fixed point, which represents the optimal solution.

Theorem 3

If T is a weakly contractive transformation in a Banach space X, then its fixed point is stable under small perturbations in φ and the space X.

Proof:

Let $T: X \to X$ be a weakly contractive transformation on a Banach space X. This means that there exists a constant $0 \leq c < 1$ such that for all $x, y \in X$,

$$d(T(x), T(y)) \leq c \cdot d(x, y).$$

Let x^* be the fixed point, so that

$$T(x^*) = x^*.$$

Now consider a small perturbation, denoted T', which is close to T in the sense that

$$d(T(x), T'(x)) \leq \epsilon \cdot d(x, y)$$

For some small $\epsilon > 0$. This perturbation T' is assumed to be weakly contractive as well, with a similar contraction constant c', where $c' < 1$.

We aim to show that the fixed point of T', denoted by x'^*, is close to the fixed point x^* of T.

Let x'^* be the fixed point of T', so that

$$T'(x'^*) = x'^*.$$

We now analyze the distance betweenx^* and x^*. By the weak contraction property, we have:

$$d\big(T(x^*), T'(x^*)\big) \leq c \cdot d(x^*, x'^*) + \epsilon \cdot d(x^*, x'^*)$$
$$= \epsilon \cdot d(x^*, x'^*).$$

Thus, the fixed points of $T$and T' are close to each other, with the distance between them shrinking as the perturbationϵ gets smaller.

Since T' is weakly contractive, it follows that x'^* will also satisfy a similar contraction property and will converge to the fixed point x^* of T as the perturbation vanishes. Specifically, for sufficiently small ϵ, the fixed point x^* of T' will be within a small neighborhood of x^*.

Thus, the fixed point x^* of $T$is stable under small perturbations in T and the Banach space X. As T approaches, its fixed point x^* approaches x'^*, demonstrating the stability of x^*

Example

Consider the compact metric space $X = [0,1]$ with the standard Euclidean metric $|d(x, y) =| x - y |$. Define a semigroup S of transformations $\{T_t\}$ on X as follows:

$$T_t(x) = \frac{x}{2} + \frac{t}{4}, for\ t \geq 0.$$

Define $g(x) = x^2$, and take $c = \frac{1}{2}$, satisfying the hybrid contractive condition

$$d\left(g\big(T_t(x)\big), g\big(T_t(y)\big)\right) = \left|\left(\frac{x}{2} + \frac{t}{4}\right)^2 - \left(\frac{y}{2} + \frac{t}{4}\right)^2\right|$$
$$\leq \frac{1}{2}|x - y|$$

Since X is compact and each T_t is hybrid contractive, there is a unique hybrid fixed point in X, given by solving $T_t(x^*) = x^*$, which leads to $x^* = \frac{x^*}{2} + \frac{t}{4}$, and hence $x^* = \frac{t}{2}$ which converges to a unique limit.

Summary and Conclusion

In conclusion, this paper provides an extension of classical fixed point theory to semigroups, with particular emphasis on the concept of hybrid fixed points. Traditionally, fixed point theorems have been a cornerstone in various mathematical fields, especially in areas involving iterative methods and optimization.

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