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The Health Economics of Life Expectancy: Counting the Days before the Individual's Roll Call

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Abstract

The precise estimation of continuous life expectancies and annuities is critical for accurate pricing and risk management in pension funds, insurance and other financial products. Gradshteyn and Ryzhik provides rigorous mathematical tools that can yield highly accurate estimates, reducing the chance of errors in approximate calculations. This can be especially beneficial when dealing with intricate mortality and discounting models. Continuous life annuities and expectancies often involve the integration of survival probabilities over a specified time horizon combined with discounting factors. These integrals can become complex but Gradshteyn and Ryzhik provides established framework which allow actuaries and financial analysts to evaluate such expressions efficiently. The objectives of the study are to obtain numerical estimation of individual's time remaining through parameterization from Makeham's law and derive closed form expressions for the complete life expectancy using the properties of the special mathematical functions as applicable in classical mortality dynamics. To circumvent the tedious estimation process associated with intractable benefit integral functions, the advanced technique of the Gradshteyn and Ryzhik's analytic integral function was deployed in computing complete life expectancies. Computational evidence showed that the males' life expectancies were less than the females' life expectancies. Within the interval $111 \le x \le 119$, there was a sharp decrease in complete life expectancies for females whereas within the interval $75 \le x \le 119$, life expectancies sharply decreased for males confirming earlier exposure of males to mortality risk.

Keywords: Makeham's law, life expectancies, generalized mortality law, Gradshteyn and Ryzhik's integral, gamma functions, parsimonious intensities

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Introduction

Life Expectancies

Life expectancy as the most useful measure of longevity represents the average number of years of life lived by a person who has survived to a defined age. Technically speaking, life expectancy defines the average value along the curve of death trajectories and the average value is most meaningful when the curve of death is symmetric. There is restricted actuarial literature which conducts full investigation of the expectations through parametrically parsimonious models [1]. This is because parsimonious studies which capture the degree of longevity in policy framework throughout entire life span have not been identified. Following observations in Palo [2], life expectancy at an age represents the mean remaining lifespan of the population the mortality data defines. However, under a current demographic mortality table, life expectancy from birth in a defined insured population represents the average total lifespan or equivalently average age at death. The estimation of life expectancy is associated with the construction of a life table to determine the proportion of life at each age x, a process which involves rigorous and exhaustive mortality data requirements and computational time. Life expectancy $\bar{e}_x$ is a useful tool in studying changes in population and it is a common mathematical summary of the average remaining lifetime a life is expected to survive from his existing age. The rationale behind its computation is associated with evaluating government's performance in improving the socio-security welfare of the population, health status and demographic projections. In mortality literature, the Gompertz $GM(x;0,2)$ life expectancy was obtained in Pflaumer [1]. Following methodology in Palo [2], the actuarial expressions developed for life expectancy expression by Missov and Lennart [3] (formula 4, page 3) seems to be inadequately expressed. This inadequacy has created opportunities to address the problem through Gradshteyn and Ryzhik's analytic integral. According to observations in Kulinskaya *et al.* [4], when addressing a number of welfare issues such as social security and health insurance schemes, life expectancy is a good device to use. Life expectancy represents a measure of mortality indicator at each age and permits comparison of mortality or longevity across geographic locations.

According to the author in [5], the expectation of life at a specified age is defined as the average of the reciprocal of the age-specific mortality intensities weighted by mortality table deaths at each age. In calculating life expectancies, the normal process is to generate a life table as a tabular statistical measure of survival and mortality experience of a cohort. In order to differentiate between mortality procedures and their corresponding effect on life expectancies, analytical techniques and continuous mortality formulations such as $GM(x;m,n)$ class are used to compute life expectancies. For a cohort whose mortality intensity is governed by $GM(x;m,n)$ force of mortality, compact expression can be formulated for the life expectancy integral associated with homogeneous populations. The essence is to create a functional framework to estimate life expectancy as applicable in population dynamics. The authors in [1, 6] obtained the complete life expectancy

$$\bar{e}_x = \frac{l_x}{K} \times \Gamma\left(0, \frac{B}{K} e^{Kx}\right) \quad (1)$$

from

$$GM(x;0,2) = Be^{Kx} \quad (1a)$$

where B is the initial mortality and K is the exponential ageing parameter

In references [1,6,7 and 16],

$$E_m(x) = x^{m-1}\Gamma(1-m, x); m = 0,1,2,3,...; x > 0 \quad (1b)$$

and consequently,

$$E_1(x) = \Gamma(0, x) \quad (1c)$$

$$E_2(x) = x\Gamma(-1, x) \quad (1d)$$

Following analytical arguments by the author in [7],

$$\Gamma(\alpha+1, x) = \alpha \times \Gamma(\alpha, x) + x^{\alpha} e^{-x} \quad (1e)$$

and for all α . If $\alpha = -1$ then

$$E_2(x) = e^{-x} - x\Gamma(0, x) = e^{-x} - xE_1(x) \quad (1f)$$

$$\Gamma(0, X) = \int_X^{\infty} \frac{e^{\xi}}{\xi} d\xi = -0.5772211566 + \log_e \frac{1}{X} + X - \frac{X^2}{4} + \frac{X^3}{18} - \frac{X^4}{96} + \frac{X^5}{600} - \frac{X^6}{4320} \quad (1g)$$

A shortcoming of the methodology in [1] is that, in an attempt to expressing both l_x and $\frac{B}{K}$ in terms of the modal age at death x_M so that $\bar{e}_x = \frac{1}{l_x}\int_x^{\Omega-x} l_u du$ can be written in the form $\int_x^{\infty} \frac{e^{-\xi}}{\xi} d\xi = \Gamma(0, x) = E_1(x)$, the author truncated the above series after the quadratic term to compute complete life expectancy without accounting for the truncation error. Given the modal age at death for human being x_M , it is clear that $e^{-K\times(x_M)} \to 0$. But recall $\frac{B}{K} = e^{-K\times(x_M)}$ and if $e^{-K\times x_M}$ were truly zero, then $\frac{B}{K}$ must also be zero. Consequently $E_1\left(\frac{B}{K}e^{Kx}\right) \to E_1(0)$ will be undefined from the above exponential integral series. This pitfall observed in [1] accounts for the reason why the analytical approach is chosen by imposing the Gradshteyn and Ryzhik's integral on $GM(1,2)$

The Generalised Makeham's Frameworks of the Committee of Mortality Investigation (CMI)

In actuarial practice, mortality models require single year age specific death rates to compute premiums for the insured. However, many published mortality tables are prepared under five-year intervals $0-4; 5-9; 10-14;...$ Therefore, a parsimonious mortality model such as $GM(1,2)$ needs be investigated to precisely generate life tables at each age and produce smooth curves which explain the underlying mortality trend. Mortality information of population is aggregated in life tables that serve as the basis for calculation of life expectancy and other life disparity measures. The CMI mortality graduation model is a complete generalization of the Makeham's law of mortality $\mu_x = \rho + GH^x$ defined in [8, 9, 10, 5, 6]. The CMI mortality function is given by

$$GM(x;m,n) = \sum_{k=0}^{m-1} a_k x^k + e^{\sum_{l=0}^{n-1} b_l x^l} \quad (1h)$$

and the logit function is given by

$$\log it(GM(x;m,n)) = \frac{\sum_{k=0}^{m-1} a_k x^k + e^{\sum_{l=0}^{n-1} b_k x^k}}{1 + \sum_{k=0}^{m-1} a_k x^k + e^{\sum_{l=0}^{n-1} b_k x^k}} \quad (1i)$$

The function $\sum_{k=0}^{m-1} a_k x^k + e^{\sum_{l=0}^{n-1} b_k x^k}$ is subject to the condition that if $m=0$, $GM(x;0,n)=e^{\sum_{l=0}^{n-1} b_k x^k}$ and if $n=0$, $GM(x;m,0)=\sum_{k=0}^{m-1} a_k x^k$ (1j)

The $GM(x;m,n)$ class is compatible with the particular set of orthogonal polynomial used in Chebyshev polynomials of the first type defined by $T_k(\cos\theta)=\cos k\theta$ so that

$T_0(x)=\cos 0=1$; $T_1(x)=x$ and $T_{1+m}(x)=2xT_m(x)-T_{m-1}(x); m\geq 1$.

The continuous form of orthogonal polynomials associated with a specified interval $a\leq x\leq b$ is determined by enumerating a sequence of polynomials $f_k(x)$; $k=0,1,2,3,...$ satisfying the following condition formulated in [9]

$$\int_a^b \varpi(x) f_m(x) f_n(x) dx = \begin{cases} 0 & m\neq n \\ \varepsilon_m & m=n \end{cases} \quad (1k)$$

where $\varpi(x)\times\sqrt{(1-x^2)}=1$ is a weight function defined on $a\leq x\leq b$ and $\varepsilon_m\in\mathbf{R}^+$

Following [9], the force of mortality can now be defined as

$$GM(x;m,n)=\sum_{k=0}^{m-1} a_k T_k(\bar{x})+e^{\sum_{l=0}^{n-1} b_l T_k(\bar{x})} \quad (1l)$$

$$GM(x;m,n)=\sum_{k=0}^{m-1} a_k T_k\left(\frac{x-p}{q}\right)+e^{\sum_{l=0}^{n-1} b_l T_k\left(\frac{x-p}{q}\right)} \quad (1m)$$

$p=\frac{1}{2}(\max x+\min x)$ and $q=\frac{1}{2}(\max x-\min x)$

and consequently, $\min x\leq x\leq \max x$ implies

$$-1\leq x\leq 1$$

Numerical Computation of the $GM(1,2)$ Parameters

In [11], the $GM(1,2)$ is defined as

$$\mu_x=\rho+GH^x \quad (1n)$$

Let $\zeta=e^{\rho}$ and $G=-\log_e\delta\log_e H$, $\zeta>0$ and $\delta>0$

The right-hand side must be multiplied by (-1) throughout by definition of the force of mortality

$$\mu_x=-\log_e\zeta+(-\log_e\delta\log_e H)H^x \quad (2)$$

$$\mu_x=-\frac{1}{l_x}\frac{dl_x}{d_x}=-\frac{d\log_e l_x}{dx} \quad (3)$$

$$\mu_x=-\frac{d\log_e l_x}{dx}=-\log_e\zeta+(-\log_e\delta\log_e H)H^x \quad (4)$$

Taking K as the constant of integration,

$$-\int\frac{d\log_e l_x}{dx}dx=\int -\log_e\zeta+(-\log_e\delta\log_e H)H^x dx+K \quad (5)$$

$$\log_e l_x = x\log_e\zeta+(\log_e\delta\log_e H)\frac{H^x}{\log_e H}+\log_e\lambda \quad (6)$$

$$K=\log_e\lambda$$

$$\log_e l_x=\log_e\zeta^x+(\log_e\delta\log_e H)\frac{H^x}{\log_e H}+\log_e\lambda \quad (7)$$

$$\log_e l_x=\log_e\zeta^x+(\log_e\delta)H^x+\log_e\lambda \quad (8)$$

where $\log_e\lambda$, is the constant of integration.

$$\log_e l_x=\log_e\zeta^x+\left(\log_e\delta^{H^x}\right)+\log_e\lambda=\log_e\lambda\zeta^x\delta^{H^x} \quad (9)$$

Now, equating both sides, we have

$$l_x=\lambda\zeta^x\delta^{H^x}\Rightarrow\int_0^\infty l_{x+s}\mu_{x+s}ds=\lambda\zeta^x\delta^{H^x} \quad (10)$$

Note that the age of the insured is chronological. We can take four of such age with equal intervals at the points $\{x+0,x+s,x+2s,x+3s\}$ to have four systems of simultaneous equations

$$l_{x+s}=\lambda\zeta^{x+s}\delta^{H^{x+s}} \quad (11)$$

$$l_{x+2s}=\lambda\zeta^{x+2s}\delta^{H^{x+2s}} \quad (12)$$

$$l_{x+3s}=\lambda\zeta^{x+3s}\delta^{H^{x+3s}} \quad (13)$$

$${}_s p_x=\frac{l_{x+s}}{l_x}=\frac{\lambda\zeta^{x+s}\delta^{H^{x+s}}}{\lambda\zeta^x\delta^{H^x}}=\frac{\zeta^s\delta^{H^{x+s}}}{\delta^{H^x}}=\zeta^s\delta^{H^{x}(H^s-1)} \quad (14)$$

Considering 4 consecutive values of function $\log_e l_x$

$$\log_e l_{x+0} = x\log_e \zeta + (\log_e \delta)H^{x} + \log_e \lambda \quad (15)$$

$$\log_e l_{x+s} = (x+s)\log_e \zeta + (\log_e \delta)H^{x+s} + \log_e \lambda \quad (16)$$

$$\log_e l_{x+2s} = (x+2s)\log_e \zeta + (\log_e \delta)H^{x+2s} + \log_e \lambda \quad (17)$$

$$\log_e l_{x+3s} = (x+3s)\log_e \zeta + (\log_e \delta)H^{x+3s} + \log_e \lambda \quad (18)$$

$$\log_e l_{x+s} - \log_e l_x = (x+s)\log_e \zeta + (\log_e \delta)H^{x+s} + \log_e \lambda$$
$$-x\log_e \zeta + (\log_e \delta)H^{x} + \log_e \lambda \quad (19)$$

$$\log_e l_{x+s} - \log_e l_x = (s\log_e \zeta) + (\log_e \delta)H^{s}H^{x} - (\log_e \delta)H^{x} \quad (20)$$

$$\log_e l_{x+s} - \log_e l_x = s\log_e \zeta + H^{x}(H^{s}-1)\log_e \delta \quad (21)$$

$$\log_e l_{x+2s} - \log_e l_{x+s} = s\log_e \zeta + H^{x+s}(H^{s}-1)\log_e \delta \quad (22)$$

$$\log_e l_{x+3s} - \log_e l_{x+2s} = s\log_e \zeta + H^{x+2s}(H^{s}-1)\log_e \delta \quad (23)$$

$$\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = (x+2s)\log_e \zeta + (\log_e \delta)H^{x+2s} + \log_e \lambda -$$
$$2\left[(x+s)(\log_e \zeta) + (\log_e \delta)H^{x+s} + \log_e \lambda\right] + x(\log_e \zeta) + (\log_e \delta)H^{x} + (\log_e \lambda) \quad (24)$$

$$\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = x\log_e \zeta + 2s\log_e \zeta + (\log_e \delta)H^{x+2s} + \log_e \lambda$$
$$-2x\log_e \zeta - 2s\log_e \zeta - 2(\log_e \delta)H^{x+s} - 2\log_e \lambda + x\log_e \zeta + (\log_e \delta)H^{x} + \log_e \lambda \quad (25)$$

$$\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = (\log_e \delta)H^{x+2s} - 2(\log_e \delta)H^{x+s} + (\log_e \delta)H^{x} \quad (26)$$

$$\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = (\log_e \delta)H^{x}\left[H^{2s} - 2H^{s} + 1\right] \quad (27)$$

Let $U = H^{s}$, then

$$\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = (\log_e \delta)U\left[U^{2} - 2U^{s} + 1\right] \quad (28)$$

$$\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = (\log_e \delta)U(U-1)^{2} \quad (29)$$

$$\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = (\log_e \delta)H^{x}(H^{x}-1)^{2} \quad (30)$$

$$\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s} = (x+3s)\log_e \zeta + (\log_e \delta)H^{x+3s} + \log_e \lambda$$
$$-2\left[(x+2s)\log_e \zeta + (\log_e \delta)H^{x+2s} + \log_e \lambda\right] + (x+s)\log_e \zeta + (\log_e \delta)H^{x+s} + \log_e \lambda \quad (31)$$

$$\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s} = x\log_e \zeta + 3s\log_e \zeta + (\log_e \delta)H^{x+3s} + \log_e \lambda$$
$$-2x\log_e \zeta - 4s\log_e \zeta - 2(\log_e \delta)H^{x+2s} - 2\log_e \lambda + x\log_e \zeta + s\log_e \zeta + (\log_e \delta)H^{x+s}$$
$$+\log_e \lambda \quad (32)$$

$$\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s} = (\log_e \delta)H^{x+3s} - 2(\log_e \delta)H^{x+2s} + (\log_e \delta)H^{x+s} \quad (33)$$

$$\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s} = (\log_e \delta)H^{x+s}\left[H^{2s} - 2H^{s} + 1\right] \quad (34)$$

$$\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s} = H^{x+s}(H^{s}-1)^{2}\log_e \delta \quad (35)$$

$$\frac{\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s}}{\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x} = \frac{= H^{x+s}(H^{s}-1)^{2}\log_e \delta}{H^{x}(H^{s}-1)^{2}\log_e \delta} = H^{x} \quad (36)$$

Since l_x values are obtained from the continuous registration system, then we let

$$\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x = \alpha \quad (37)$$

$$\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s} = \beta \quad (38)$$

$$H^x\left(H^s - 1\right)^2 \log_e \delta = \alpha \quad (39)$$

$$H^{x+s}\left(H^s - 1\right)^2 \log_e \delta = \beta \quad (40)$$

Taking logarithms of the two equations above, we have

$$x\log_e H + 2\log_e\left(H^s - 1\right) + \log_e \log_e \delta = \log_e \alpha \quad (41)$$

$$(x+s)\log_e H + 2\log_e\left(H^s - 1\right) + \log_e \log_e \delta = \log_e \beta \quad (42)$$

Subtracting equation (41) from(42), we obtain

$$x\log_e H + s\log_e H + 2\log\left(H^s - 1\right) + \log_e \log_e \delta - x\log_e H - 2\log_e\left(H^s - 1\right) - \log_e \log_e \delta = \log_e \beta - \log_e \alpha \quad (43)$$

$$s\log_e H = \log_e \beta - \log_e \alpha \quad (44)$$

$$\log_e H = \frac{\log_e \beta - \log_e \alpha}{s} = \frac{\log_e \frac{\beta}{\alpha}}{s} \quad (45)$$

$$x\log_e H + 2\log_e\left(H^s - 1\right) + \log_e \log_e \delta = \log_e \alpha \quad (46)$$

substitute (45) in (41)

$$\frac{x}{s}\log_e \frac{\beta}{\alpha} + 2\log_e\left(H^s - 1\right) + \log_e \log_e \delta = \log_e \alpha \quad (47)$$

$$\log_e \log_e \delta = \log_e \alpha - \frac{x}{s}\log_e \frac{\beta}{\alpha} - 2\log_e\left(H^s - 1\right) \quad (48)$$

$$\log_e\left[\log_e \delta\right] = \log_e \alpha + \log_e\left(\frac{\beta}{\alpha}\right)^{-\frac{x}{s}} + \log_e\left(H^s - 1\right)^{-2} = \log_e\left(\frac{\beta}{\alpha}\right)^{-\frac{x}{s}}\left(H^s - 1\right)^{-2} \quad (49)$$

Equation (49) then becomes

$$\log_e \delta = \alpha\left(\frac{\beta}{\alpha}\right)^{-\frac{x}{s}}\left(H^s - 1\right)^{-2} \quad (50)$$

$$x\log_e H + 2\log_e\left(H^s - 1\right) + \log_e \log_e \delta = \log_e \alpha \quad (51)$$

in (46) is re-expressed as

$$x\log_e H + \log_e\left(H^s - 1\right) + \log_e \log_e \delta = \log_e \alpha - \log_e\left(H^s - 1\right) \quad (52)$$

$$\log_e\left[H^x\left(H^s - 1\right)\log_e \delta\right] = \log_e \frac{\alpha}{\left(H^s - 1\right)} \quad (53)$$

$$H^x\left(H^s - 1\right)\log_e \delta = \frac{\alpha}{\left(H^s - 1\right)} \quad (54)$$

$$\log_e l_{x+s} - \log_e l_x = s\log_e \zeta + H^x\left(H^s - 1\right)\log_e \delta \quad (55)$$

Substituting equation (54) in (55), we have

$$\log_e l_{x+s} - \log_e l_x = s\log_e \zeta + \frac{\alpha}{\left(H^s - 1\right)} \quad (56)$$

$$s\log_e \zeta = \log_e l_{x+s} - \log_e l_x - \frac{\alpha}{\left(H^s - 1\right)} \quad (57)$$

$$\log_e \zeta = \left[\frac{\log_e l_{x+s} - \log_e l_x - \frac{\alpha}{(H^s - 1)}}{s} \right] \quad (58)$$

Recall from (52) that,

$$x \log_e H + \log_e (H^s - 1) + \log_e \log_e \delta = \log_e \alpha - \log_e (H^s - 1) \quad (59)$$

$$x \log_e H + \log_e \log_e \delta = \log_e \alpha + \log_e (H^s - 1)^{-2} \quad (60)$$

$$\log_e [H^x \log_e \delta] = \log_e \alpha (H^s - 1)^{-2} \quad (61)$$

$$H^x \log_e \delta = \alpha (H^s - 1)^{-2} \quad (62)$$

$$H^x = \frac{\alpha}{\log_e \delta} (H^s - 1)^{-2} \quad (63)$$

again observe that

$$x \log_e \zeta + (\log_e \delta) H^x + \log_e \lambda = \log_e l_x \quad (64)$$

$$\log_e \lambda = \log_e l_x - x \log_e \zeta - (\log \delta) H^x \quad (65)$$

put (58), (63) in (64)

$$\log_e \lambda = \log_e l_x - x \left[\frac{\log_e l_{x+s} - \log_e l_x - \frac{\alpha}{(H^s - 1)}}{s} \right] - \alpha (H^s - 1)^{-2} \quad (66)$$

$$\log_e \zeta = \rho$$

$$\rho = \left[\frac{\log_e l_{x+s} - \log_e l_x - \frac{\alpha}{(H^s - 1)}}{s} \right] \quad (67)$$

and by the initial definition $G = -\log_e \delta \log_e H$

recall

$$\log_e H = \frac{\log_e \frac{\beta}{\alpha}}{s} = \log_e \left(\frac{\beta}{\alpha} \right)^{\frac{1}{s}} \quad (68)$$

$$H = \left(\frac{\beta}{\alpha} \right)^{\frac{1}{s}} \quad (69)$$

Note that

$$G = \frac{(-\log_e \delta)}{s} \log_e \frac{\beta}{\alpha} \quad (70)$$

and

$$H^x = \frac{\alpha}{\log_e \delta} (H^s - 1)^{-2} \quad (71)$$

$\mu_x = \rho + GH^x$ becomes

$$\mu_x = \left[\frac{\log_e l_{x+s} - \log_e l_x - \dfrac{\alpha}{(H^s - 1)}}{s}\right] + \left[\frac{(-\log_e \delta)}{s}\log_e \frac{\beta}{\alpha}\right]\left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \tag{72}$$

$$\mu_x = \left[\frac{\log_e \dfrac{\log_e l_{x+s}}{l_x} - \dfrac{\alpha}{(H^s - 1)}}{s}\right] + \left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \frac{(-\log_e \delta)}{s}\log_e \frac{\beta}{\alpha} \tag{73}$$

$$\mu_x = \left[\frac{\log_e ({}_sP_x) - \dfrac{\alpha}{(H^s - 1)}}{s}\right] + \left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \frac{(-\log_e \delta)}{s}\log_e \frac{\beta}{\alpha} \tag{74}$$

$$\mu_x = \left[\frac{\log_e \zeta^2 \delta^{H^x(H^s-1)} - \dfrac{\alpha}{(H^s - 1)}}{s}\right] + \left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \frac{(-\log_e \delta)}{s}\log_e \frac{\beta}{\alpha} \tag{75}$$

Recall

$$H = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{s}} \Rightarrow H^s = \left(\frac{\beta}{\alpha}\right) \tag{76}$$

So when $s = x$, we have

$$H^x = \left(\frac{\beta}{\alpha}\right) \tag{77}$$

$$\frac{\log_e l_{x+3s} - 2\log_e l_{x+2s} + \log_e l_{x+s}}{\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x} = \frac{= H^{x+s}(H^s - 1)^2 \log_e \delta}{H^x (H^s - 1)^2 \log_e \delta} = H^x = \left(\frac{\beta}{\alpha}\right) \tag{78}$$

Hence, we obtain

$$\mu_x = \left[\frac{\log_e \zeta^2 \delta^{H^x(H^s-1)} - \dfrac{\alpha}{(H^s - 1)}}{s}\right] + \left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \frac{(-\log_e \delta)}{s}\log_e \frac{(\log_e l_{x+3s} - 2\log_e l_{x+2s} \log_e l_{x+s})}{\log_e l_{x+2s} - 2\log_e l_{x+s} + \log_e l_x} \tag{79}$$

$$\mu_x = \left[\frac{\log_e \zeta^s \delta^{H^x(H^s-1)} - \dfrac{\alpha}{(H^s - 1)}}{s}\right] + \left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \frac{(-\log_e \delta)}{s}\log_e \frac{(\log_e \lambda\zeta^{x+3s}\delta^{H^{x+3s}} - 2\log_e \lambda\zeta^{x+2s}\delta^{H^{x+2s}} + \log_e \lambda\zeta^{x+s}\delta^{H^{x+s}})}{\log_e \lambda\zeta^{x+2s}\delta^{H^{x+3s}} - 2\log_e \lambda\zeta^{x+s}\delta^{H^{x+s}} + \log_e \lambda\zeta^{x}\delta^{H^{x}}} \tag{80}$$

$$\mu_x = \left[\frac{\log_e \zeta^s \delta^{H^x(H^s-1)} - \dfrac{\alpha}{(H^s-1)}}{s} \right] + \left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \frac{(-\log_e \delta)}{s} \log_e \frac{\left(\begin{array}{l} \log_e \lambda\zeta^{x+3s}\delta^{H^{x+3s}} - \log_e \lambda\zeta^{x+2s}\delta^{H^{x+2s}} + \log_e \lambda\zeta^{x+s}\delta^{H^{x+s}} \\ -\log_e \lambda\zeta^{x+2s}\delta^{H^{x+2s}} \end{array} \right)}{\begin{array}{l} \log_e \lambda\zeta^{x+2s}\delta^{H^{x+3s}} - \log_e \lambda\zeta^{x+s}\delta^{H^{x+s}} + \log_e \lambda\zeta^{x}\delta^{H^{x}} \\ -\log_e \lambda\zeta^{x+s}\delta^{H^{x+s}} \end{array}} \quad (81)$$

$$\mu_x = \left[\frac{\log_e \zeta^s \delta^{H^x(H^X-1)} - \dfrac{\alpha}{(H^s-1)}}{s} \right] + \left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \frac{(-\log_e \delta)}{s} \log_e \frac{\left(\log_e \dfrac{\lambda\zeta^{x+3s}\delta^{H^{x+3s}}}{\log_e \lambda\zeta^{x+2s}\delta^{H^{x+2s}}} + \log_e \dfrac{\lambda\zeta^{x+s}\delta^{H^{x+s}}}{\lambda\zeta^{x+2s}\delta^{H^{x+2s}}} \right)}{\log_e \dfrac{\lambda\zeta^{x+2s}\delta^{H^{x+3s}}}{\log_e \lambda\zeta^{x+s}\delta^{H^{x+s}}} + \log_e \dfrac{\log_e \lambda\zeta^{x}\delta^{H^{x}}}{\lambda\zeta^{x+s}\delta^{H^{x+s}}}} \quad (82)$$

$$\mu_x = \left[\frac{\log_e \zeta^s \delta^{H^x(H^X-1)} - \dfrac{\alpha}{(H^s-1)}}{s} \right] + \left(\frac{\beta}{\alpha}\right)^{\frac{x}{s}} \frac{(-\log_e \delta)}{s} \log_e \left[\frac{\left(\log_e \dfrac{\delta^{H^{x+3s}}}{\delta^{H^{x+2s}}} \dfrac{\delta^{H^{x+s}}}{\delta^{H^{x+2s}}} \right)}{\log_e \dfrac{\delta^{H^{x+3s}}}{\delta^{H^{x+s}}} \dfrac{\delta^{H^{x}}}{\delta^{H^{x+s}}}} \right] \quad (83)$$

THEOREM

Suppose

$$\bar{A}_x = \frac{\int_0^\infty e^{(-\delta_x - \delta_y)} l_y \mu_y dy}{\int_0^\infty l_{x+t}\mu_{x+t} dt} \quad (84a)$$

$$q_x = \frac{\frac{1}{2}(\mu_x + \mu_{x+1})}{1 + \frac{1}{2}\mu_{x+1}} \quad (84b)$$

Then

$$\frac{d}{dx}\bar{e}_x = -1 + (2q_x - p_x\mu_{x+1}) \times (1 + \bar{e}_{x+1}) \quad (85)$$

Proof

We recognize that

$$\int_0^\infty l_{x+t}\mu_{x+t} dt = l_x \quad (86)$$

$$\bar{A}_x = \frac{1}{{}_x p_0 v^x} \int_x^\infty v^y \, {}_y p_0 \mu(y) dy \quad (87)$$

$$\bar{A}_x \times {}_x p_0 e^{-\delta x} = \int_x^\infty e^{-\delta y} \, {}_y p_0 \mu(y) dy \quad (88)$$

$$\frac{d}{dx}\bar{A}_x \times {}_x P_0 V^x = \frac{d}{dx}\int_x^0 V^y \, {}_y P_0 \mu(y) dy \quad (89)$$

$$\frac{d}{dx}\left({}_x p_0 \times \bar{A}_x \times v^x\right) = \bar{A}_x v^x \frac{d}{dx}\frac{l_x}{l_0} + \bar{A}_x \frac{l_x}{l_0}\frac{d}{dx}v^x + v^x \frac{l_x}{l_0}\frac{d}{dx}\bar{A}_x \quad (90)$$

$$\frac{1}{l_0}\frac{d}{dx}\bar{A}_x l_x v^x = \frac{1}{l_0}\bar{A}_x v^x (l_x') + \frac{\bar{A}_x}{l_0} l_x \frac{d}{dx} v^x + v^x \frac{l_x}{l_0}\frac{d}{dx}\bar{A}_x \quad (91)$$

$$\frac{1}{l_0}\frac{d}{dx}\bar{A}_x l_x v^x = -\frac{1}{l_0}\bar{A}_x v^x (\mu_x l_x) + \frac{\bar{A}_x}{l_0} l_x (\log_e v) v^x + v^x \frac{l_x}{l_0}\frac{d}{dx}\bar{A}_x \quad (92)$$

$$\frac{1}{l_0}\frac{d}{dx}\bar{A}_x l_x v^x = -\frac{1}{l_0}\bar{A}_x v^x (\mu_x l_x) + \frac{\bar{A}_x}{l_0} l_x (\log_e e^{-\delta}) v^x + v^x \frac{l_x}{l_0}\frac{d}{dx}\bar{A}_x \quad (93)$$

$$\frac{1}{l_0}\frac{d}{dx}\bar{A}_x l_x v^x = -\frac{1}{l_0}\bar{A}_x v^x \mu_x l_x - \frac{1}{l_0}\bar{A}_x l_x \delta v^x + v^x \frac{l_x}{l_0}\frac{d}{dx}\bar{A}_x \quad (94)$$

Now,

$$\int_x^\infty v^y \, {}_y P_0 \, \mu_y dy = \frac{1}{l_0}\int_x^\infty v^y l_y \mu_y dy \quad (95)$$

$$\frac{1}{l_0}\frac{d}{dx}\int_x^\infty v^y l_y \mu_y dy = \frac{1}{l_0}\left[0 - v^x l_x \mu_x\right] = -\frac{1}{l_0} v^x l_x \mu_x \quad (96)$$

$$-\frac{1}{l_0}\bar{A}_x v^x \mu_x l_x - \frac{1}{l_0}\bar{A}_x l_x \delta v^x + v^x \frac{l_x}{l_0}\frac{d}{dx}\bar{A}_x = -\frac{1}{l_0} v^x l_x \mu_x \quad (97)$$

$$-\bar{A}_x \mu_x - \bar{A}_x \delta + \frac{d}{dx}\bar{A}_x = -\mu_x \quad (98)$$

$$\frac{d}{dx}\bar{A}_x = \bar{A}_x\mu_x - \mu_x + \bar{A}_x\delta \quad (99)$$

$$\frac{d}{dx}\left(1-\delta\bar{a}_x\right) = \left(1-\delta\bar{a}_x\right)\mu_x - \mu_x + \left(1-\delta\bar{a}_x\right)\delta \quad (100)$$

$$-\delta\frac{d}{dx}\bar{a}_x = -\delta\bar{a}_x\mu_x + \delta - \delta^2\bar{a}_x \quad (101)$$

$$\frac{d}{dx}\bar{a}_x = \bar{a}_x\mu_x - 1 + \delta\bar{a}_x \quad (102)$$

$$\frac{d}{dx}\lim_{\delta\to 0}\bar{a}_x = \lim_{\delta\to 0}\bar{a}_x\mu_x - \lim_{\delta\to 0}1 + \delta\lim_{\delta\to 0}\bar{a}_x \quad (103)$$

$$\frac{d}{dx}\bar{e}_x = \bar{e}_x\mu_x - 1 \quad (104)$$

Following [12-13],

$$q_x = \frac{\frac{1}{2}\left(\mu_x + \mu_{x+1}\right)}{1+\frac{1}{2}\mu_{x+1}} \quad (105)$$

$$\frac{1}{2}\mu_x = q_x + \frac{1}{2}q_x\mu_{x+1} - \frac{1}{2}\mu_{x+1} \quad (106)$$

$$\mu_x = 2q_x + q_x\mu_{x+1} - \mu_{x+1} \quad (107)$$

$$\mu_x = 2q_x + \left(q_x - 1\right)\mu_{x+1} \quad (108)$$

$$\mu_x = 2q_x - \left(1-q_x\right)\mu_{x+1} \quad (109)$$

$$\mu_x = 2q_x - p_x\mu_{x+1} \quad (110)$$

$$\frac{d}{dx}\bar{e}_x = \bar{e}_x\left(2q_x - p_x\mu_{x+1}\right) - 1 \quad (111)$$

$$\frac{d}{dx}\bar{e}_x = -1 + \left(2q_x - p_x\mu_{x+1}\right)\times\int_0^{\Omega-x}{}_{\xi}p_x d\xi \quad (112)$$

$$\frac{d}{dx}\bar{e}_x = -1 + \left(2q_x - p_x\mu_{x+1}\right)\times\left(\int_0^1{}_{\xi}p_x d\xi + \int_1^{\Omega-x}{}_{\xi}p_x d\xi\right) \quad (113)$$

$$\frac{d}{dx}\bar{e}_x \le -1 + \left(2q_x - p_x\mu_{x+1}\right)\times\left(1 + \int_1^{\Omega-x}{}_{\xi}p_x d\xi\right) \quad (114)$$

$$\frac{d}{dx}\bar{e}_x = -1 + \left(2q_x - p_x\mu_{x+1}\right)\times\left(1 + \int_1^{\Omega-x}\left({}_1p_x\right)\left({}_{\xi-1}p_{x+1}\right)d\xi\right) \quad (115)$$

$$\frac{d}{dx}\bar{e}_x \le -1 + \left(2q_x - p_x\mu_{x+1}\right)\times\left(1 + \int_1^{\Omega-x}\left({}_{\xi-1}p_{x+1}\right)d\xi\right) \quad (116)$$

$$\frac{d}{dx}\bar{e}_x \le -1 + \left(2q_x - p_x\mu_{x+1}\right)\times\left(1 + \int_0^{\Omega-x}\left({}_{\xi}p_{x+1}\right)d\xi\right) \quad (117)$$

$$\frac{d}{dx}\bar{e}_x = -1 + \left(2q_x - p_x\mu_{x+1}\right)\times\left(1 + \bar{e}_{x+1}\right) \quad (118)$$

$Q.E.D$

The Force of Interest

In life insurance valuation, it is assumed that the force of interest is constant. This force of interest is generated as follows.

$$\left(1-\frac{d^{(K)}}{K}\right)^{-K} = 1 + K\times\left(\frac{d^{(K)}}{K\left(1-d^{(K)}\right)}\right)^1 + \frac{K(K-1)}{2!}\times\left(\frac{d^{(K)}}{K\left(1-d^{(K)}\right)}\right)^2 + \frac{K(K-1)(K-2)}{3!}\times\left(\frac{d^{(K)}}{K\left(1-d^{(K)}\right)}\right)^3 + \ldots \quad (119)$$

But

$$\left(1+\frac{i^{(K)}}{K}\right) = \left(1-\frac{d^{(K)}}{K}\right)^{-1} \quad (120)$$

$$\left(1+\frac{i^{(K)}}{K}\right)^K = 1 + K\times\left(\frac{i^{(K)}}{K}\right)^1 + \frac{K(K-1)}{2!}\times\left(\frac{i^{(K)}}{K}\right)^2 + \frac{K(K-1)(K-2)}{3!}\times\left(\frac{i^{(K)}}{K}\right)^3 + \ldots \quad (121)$$

$$\left(1+\frac{i^{(K)}}{K}\right)^K = 1 + K\times\left(\frac{i^{(K)}}{K}\right)^1 + \frac{K(K-1)}{K^2}\times\frac{1}{2!}\left(i^{(K)}\right)^2 + \frac{K(K-1)(K-2)}{K^3}\times\frac{1}{3!}\left(i^{(K)}\right)^3 + \ldots \quad (122)$$

$$\lim_{K\to\infty}\left(1+\frac{i^{(K)}}{K}\right)^K = \lim_{K\to\infty}\left\{\begin{matrix}1 + K\times\left(\frac{i^{(K)}}{K}\right)^1 + \frac{K(K-1)}{K^2}\times\frac{1}{2!}\left(i^{(K)}\right)^2 + \frac{K(K-1)(K-2)}{K^3}\times\frac{1}{3!}\left(i^{(K)}\right)^3 \\ +\ldots+\ldots\end{matrix}\right\} \quad (123)$$

$$\lim_{K\to\infty}\left(1+\frac{i^{(K)}}{K}\right)^{K}=\lim_{K\to\infty}\left\{1+i^{(K)}+\frac{K^{2}\left(1-\frac{1}{K}\right)}{K^{2}}\times\frac{1}{2!}\left(i^{(K)}\right)^{2}+\frac{K^{3}\left(1-\frac{1}{K}\right)\left(1-\frac{2}{K}\right)}{K^{3}}\times\frac{1}{3!}\left(i^{(K)}\right)^{3}+\ldots\right\} \quad (124)$$

$$\lim_{K\to\infty}\left(1+\frac{i^{(K)}}{K}\right)^{K}=1+i^{(\infty)}+\frac{1}{2!}\left(i^{(\infty)}\right)^{2}+\frac{1}{3!}\left(i^{(\infty)}\right)^{3}+\frac{1}{4!}\left(i^{(\infty)}\right)^{4}\ldots \quad (125)$$

But

$$e^{\sigma}=1+\frac{\sigma}{1!}+\frac{\sigma^{2}}{2!}+\frac{\sigma^{3}}{3!}+\frac{\sigma^{4}}{4!}\ldots \quad (126)$$

Consequently,

$$\lim_{K\to\infty} i^{(K)}=\log_{e}(1+i)=\sigma \quad (127)$$

where $\sigma:[0,\infty)\to\mathbf{R}^{+}$ is the interest rate intensity.

Based on equations 119-127, the following consequence are immediate: T_x be the complete future random life time. Assume that

$$\Sigma^{n}=\left(\frac{1}{n},\frac{2}{n},\frac{3}{n},\ldots,\frac{n-1}{n},\frac{n}{n}\right) \quad (127a)$$

and suppose that the monthly payment disbursed to a retiree be defined as

$$\gamma^{n}=\frac{1}{n}\left(1+v^{\frac{1}{n}}+v^{\frac{2}{n}}+v^{\frac{3}{n}}+\ldots+v^{T_x+\Sigma^{n}-\frac{1}{n}}\right) \quad (127b)$$

$$\gamma^{n}=\frac{1}{n}\left[\left(v^{\frac{1}{n}}\right)^{0}+\left(v^{\frac{1}{n}}\right)^{1}+\left(v^{\frac{1}{n}}\right)^{2}+\left(v^{\frac{1}{n}}\right)^{3}+\ldots+\left(v^{\frac{1}{n}}\right)^{nT_x+n\Sigma^{n}-1}\right] \quad (127c)$$

$$\gamma^{n}=\frac{1}{n}\left[\frac{1-v^{T_x+\Sigma^{n}}}{1-v^{\frac{1}{n}}}\right];v=\frac{1}{1+i} \quad (127d)$$

Now taking expectation we have

$$\mathbf{E}\left(\gamma^{n}\right)=\mathbf{E}\left(\frac{1}{n}\left[\frac{1-v^{T_x+\Sigma^{n}}}{1-v^{\frac{1}{n}}}\right]\right) \quad (127e)$$

assuming that Σ^{n} and T_n are independent.

$$\ddot{a}_{x}^{(n)}=\frac{1}{n}\left(\frac{1}{1-v^{\frac{1}{n}}}\right)E\left(1-v^{T_x+\Sigma^{n}}\right)=\frac{1}{n}\left(\frac{1}{1-v^{\frac{1}{n}}}\right)\left[1-\mathbf{E}\left(v^{T_x}\right)\mathbf{E}\left(v^{\Sigma^{n}}\right)\right]=\frac{1}{n}\left(\frac{1}{1-v^{\frac{1}{n}}}\right)\left[1-\bar{A}_x\mathbf{E}\left(v^{\Sigma^{n}}\right)\right] \quad (127f)$$

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$$\int_{0}^{\tau}\mu_{x+\alpha}(\xi)d\xi=-\left[\log_{e}S_{x}(\alpha,\xi)\right]_{0}^{\tau} \quad (128)$$

$$\mu_{x}(\xi)\times\left(\int_{0}^{\tau}d\alpha\right)=\left[\log_{e}S_{x}(0,\xi)-\log_{e}S_{x}(\tau,\xi)\right] \quad (128a)$$

$$\left[\log_{e}1-\log_{e}S_{x}(\tau,\xi)\right]=\tau\times\mu_{x}(\xi) \quad (128b)$$

$$S_{x}(\tau,\xi)=e^{-\tau\times\mu_{x}(\xi)} \quad (128c)$$

$$\int_{0}^{1}S_{x}(\alpha,\xi)d\alpha=\int_{0}^{1}e^{-\alpha\times\mu_{x}(\xi)}d\alpha \quad (128d)$$

$$\int_{0}^{1}S_{x}(\alpha,\xi)d\alpha=\left[\frac{e^{-\alpha\times\mu_{x}(\xi)}}{\mu_{x}(\xi)}\right]_{\alpha=0}^{1} \quad (128e)$$

$$\int_{0}^{1}S_{x}(\alpha,\xi)d\alpha=\left[\frac{\left(1-e^{-\mu_{x}(\xi)}\right)}{\mu_{x}(\xi)}\right] \quad (128f)$$

$$S_{x}(\alpha,\xi)=\prod_{k=0}^{\alpha-1}S_{x+k}(1,\xi) \quad (128g)$$

$$S_{x}(\alpha,\xi)=\prod_{k=0}^{\alpha-1}\left(p_{x+k}(\xi)\right) \quad (128h)$$

$$e_{x}(\xi)=\int_{0}^{\infty}S_{x}(\alpha,\xi)d\alpha \quad (128i)$$

$$e_{x}(\xi)=\int_{0}^{1}S_{x}(\alpha,\xi)d\alpha+\sum_{k=1}^{\Omega-1}\left[S_{x}(k,\xi)\int_{0}^{1}S_{x}(k+\alpha,\xi)d\alpha\right] \quad (128j)$$

$$e_{x}(\xi)=\frac{\left(1-e^{-\mu_{x}(\xi)}\right)}{\mu_{x}(\xi)}+\sum_{k=1}^{\Omega-1}\prod_{\alpha=0}^{k-1}\left[\frac{\left(p_{x+\alpha}(\xi)\right)\times\left(1-e^{-\mu_{x+k}(\xi)}\right)}{\mu_{x+k}(\xi)}\right] \quad (128k)$$

Material and Methods

Following [8, 14, 3,15], the continuous whole life annuity is defined as

$$\mathbf{E}\left(a_{\overline{T}|}\right)=\bar{a}_{x}=\frac{1}{l_x}\int_{0}^{\Omega-x}v^{\xi}l_{x+\xi}d\xi \quad (128l)$$

where Ω is the maximum age in life tables

$$\bar{a}_{x}=\int_{0}^{\Omega-x}\left(e^{\sigma}\right)^{(-\xi)}\left(\frac{l_{x+\xi}}{l_x}\right)d\xi \quad (129)$$

$$\bar{a}_{x}=\int_{0}^{\Omega-x}e^{-\sigma\xi}\left({}_{\xi}p_{x}\right)d\xi \quad (130)$$

$$\bar{a}_{x}=\int_{0}^{\Omega-x}\zeta^{\xi}\delta^{H^{x}\left(H^{\xi}-1\right)}e^{-\sigma\xi}d\xi \quad (131)$$

$$ {}_{\xi}p_x e^{-\sigma\xi} = e^{-\sigma\xi}\zeta^{\xi}\delta^{H^x\left(H^{\xi}-1\right)} = e^{-\sigma\xi}\zeta^{\xi}\delta^{\left(H^{x+\xi}-H^x\right)} = \frac{e^{-\sigma\xi}\zeta^{\xi}\delta^{\left(H^{x+\xi}\right)}}{\delta^{H^x}} \quad (132) $$

$$ {}_{\xi}p_x e^{-\sigma\xi} = e^{-\sigma\xi}\zeta^{\xi}\delta^{H^x\left(H^{\xi}-1\right)} = \frac{e^{-\sigma\xi}\zeta^{\xi}\delta^{\left(H^{x+\xi}\right)}}{\delta^{H^x}} = \frac{\exp\left(H^{\xi}H^x\log_e\delta + \xi\log_e\left(e^{-\sigma}\zeta\right)\right)}{\delta^{H^x}} \quad (133) $$

Observe that

$$ H^{\xi} = e^{\log_e H^{\xi}} = e^{\xi\log_e H} \quad (134) $$

$$ {}_{\xi}p_x e^{-\sigma\xi} = e^{-\sigma\xi}\zeta^{\xi}\delta^{H^x\left(H^{\xi}-1\right)} = \frac{e^{-\sigma\xi}\zeta^{\xi}g^{\left(H^{x+\xi}\right)}}{\delta^{H^x}} = \frac{\exp\left(\left(e^{\xi\log_e H}\right)\left(H^x\times\log_e g\right)+\xi\log_e e^{-\sigma}\zeta\right)}{\delta^{H^x}} \quad (135) $$

$$ \bar{a}_x = \int_0^{\Omega-x} \frac{\exp\left(\left(e^{\xi\log_e H}\right)\left(H^x\times\log_e\delta\right)+\xi\log_e e^{-\sigma}\zeta\right)}{\delta^{H^x}} d\xi \quad (136) $$

$$ \text{Let } \eta = \xi\log_e H \Rightarrow \frac{\eta}{\log_e H} = \xi \quad (137) $$

when $\xi = 0$, $\eta = 0$

when $\xi = \Omega - x$,

$$ \eta = \left(\Omega - x\right)\log_e H = \log_e H^{(\Omega-x)} \quad (138) $$

$$ \frac{d\eta}{dt} = \log_e H \Rightarrow d\eta = \log_e H d\xi \Rightarrow \frac{d\eta}{\log_e H} = d\xi \quad (139) $$

$$ \bar{a}_x = \int_0^{\log_e\left(H^{(\Omega-x)}\right)} \left[\frac{\exp\left(e^{\eta}\left(H^x\times\log_e\delta\right)+\frac{\eta}{\log_e H}\times\log_e e^{-\sigma}\zeta\right)}{\delta^{H^x}}\right]\frac{d\eta}{\log_e H} \quad (140) $$

$$ \bar{a}_x = \frac{1}{\left(\log_e H\right)\left(\delta^{H^x}\right)} \int_0^{\log_e H^{(\Omega-x)}} \left[\exp\left(\left(e^{\eta}\times H^x\times\log_e\delta\right)+\frac{\eta}{\log_e H}\times\log_e e^{-\sigma}\zeta\right)\right]d\eta \quad (141) $$

$$ \bar{a}_x = \frac{1}{\left(\log_e H\right)\left(\delta^{H^x}\right)} \int_0^{\log_e H^{(\Omega-x)}} \left[\exp\left(\left(H^x\times\log_e\delta\right)e^{\eta}+\left(\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)\eta\right)\right]d\eta \quad (142) $$

Using the [16-17], the continuous life annuity is obtained as

$$ \bar{a}_x = \frac{1}{\left(\log_e H\right)\left(\delta^{H^x}\right)} \left\{ \begin{array}{l} \left[\left(-H^x\times\log_e\delta\right)\right]^{\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)}\Gamma\left(\left(\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right),\left(-H^x\times\log_e\delta\right)\right) - \\ e^{-\sigma(\Omega-x)}\left[\left(-\log_e\delta\right)H^{\Omega}\right]^{\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)}\Gamma\left(\left(\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right),\left(-\log_e\delta\right)H^{\Omega}\right) \end{array} \right\} \quad (143) $$

Taking the limit as the force of interest approach zero, we have the life expectancy

$$ \lim_{\sigma\to 0}\bar{a}_x = \lim_{\sigma\to 0}\int_0^{\infty} e^{-\sigma\xi}\,{}_{\xi}p_x d\xi = \int_0^{\infty}{}_{\xi}p_x d\xi = \bar{e}_x \quad (144) $$

$$ \bar{e}_x = \frac{1}{\left(\log_e H\right)\left(\delta^{H^x}\right)} \lim_{\sigma\to 0} \left\{ \begin{array}{l} \left[\left(-H^x\times\log_e\delta\right)\right]^{\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)}\Gamma\left(\left(\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right),\left(-H^x\times\log_e\delta\right)\right) - \\ e^{-\sigma(\Omega-x)}\left[\left(-\log_e\delta\right)H^{\Omega}\right]^{\left(-\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right)}\Gamma\left(\left(\frac{\log_e e^{-\sigma}\zeta}{\log_e H}\right),\left(-\log_e\delta\right)H^{\Omega}\right) \end{array} \right\} \quad (145) $$

For ease of computation, equation (145) was arbitrarily broken down into segments as shown in columns $1-10$ and in column 11 the continuous life expectancy is finally computed.

Data Analysis and Presentation

Having derived the life expectancy formulae based on $\mu_x = \rho + GH^x$, 2020 current survival data from the US social security tables were used to estimate the male parameters using advanced algebraic method as $\rho = 0.0006967734552$; $G = 0.0000344941056$ and $H = 1.099250547$

Table 1 below shows the systematic computation of life expectancies based on equation (145) which has been broken down systematically into columns $2-10$ and finally computed in column 11.

Table 1: Male life expectancy $\bar{e}_x$

Age	2	3	4	5	6	7	8	9	10	11
0	0.094629	0.000035	0.000697	0.000000	1.000000	0.007363	10.645730	0.964483	4.424266	45.426710
1	0.094629	0.000035	0.000697	0.094629	1.099251	0.008094	10.653510	0.965156	4.326942	44.490880
2	0.094629	0.000035	0.000697	0.189257	1.208352	0.008897	10.662070	0.965828	4.229760	43.556920
3	0.094629	0.000035	0.000697	0.283886	1.328281	0.009780	10.671490	0.966501	4.132727	42.624990
4	0.094629	0.000035	0.000697	0.378515	1.460114	0.010751	10.681850	0.967175	4.035851	41.695280
5	0.094629	0.000035	0.000697	0.473143	1.605031	0.011818	10.693260	0.967849	3.939142	40.768000
6	0.094629	0.000035	0.000697	0.567772	1.764331	0.012991	10.705810	0.968524	3.842608	39.843350
7	0.094629	0.000035	0.000697	0.662400	1.939442	0.014281	10.719620	0.969199	3.746260	38.921560
8	0.094629	0.000035	0.000697	0.757029	2.131933	0.015698	10.734830	0.969875	3.650109	38.002870
9	0.094629	0.000035	0.000697	0.851658	2.343528	0.017256	10.751560	0.970551	3.554168	37.087510
10	0.094629	0.000035	0.000697	0.946286	2.576125	0.018969	10.769990	0.971227	3.458450	36.175760
11	0.094629	0.000035	0.000697	1.040915	2.831807	0.020851	10.790290	0.971904	3.362970	35.267890
12	0.094629	0.000035	0.000697	1.135544	3.112865	0.022921	10.812640	0.972581	3.267745	34.364180
13	0.094629	0.000035	0.000697	1.230172	3.421819	0.025196	10.837270	0.973259	3.172793	33.464950
14	0.094629	0.000035	0.000697	1.324801	3.761436	0.027696	10.864400	0.973938	3.078133	32.570500
15	0.094629	0.000035	0.000697	1.419429	4.134760	0.030445	10.894310	0.974617	2.983787	31.681180
16	0.094629	0.000035	0.000697	1.514058	4.545138	0.033467	10.927280	0.975296	2.889778	30.797320
17	0.094629	0.000035	0.000697	1.608687	4.996245	0.036789	10.963630	0.975976	2.796132	29.919280
18	0.094629	0.000035	0.000697	1.703315	5.492125	0.040440	11.003740	0.976656	2.702876	29.047450
19	0.094629	0.000035	0.000697	1.797944	6.037222	0.044454	11.047990	0.977337	2.610042	28.182210
20	0.094629	0.000035	0.000697	1.892573	6.636419	0.048866	11.096840	0.978018	2.517661	27.323960
21	0.094629	0.000035	0.000697	1.987201	7.295087	0.053715	11.150790	0.978700	2.425772	26.473120
22	0.094629	0.000035	0.000697	2.081830	8.019129	0.059047	11.210400	0.979382	2.334412	25.630120
23	0.094629	0.000035	0.000697	2.176458	8.815032	0.064907	11.276290	0.980064	2.243625	24.795410
24	0.094629	0.000035	0.000697	2.271087	9.689928	0.071349	11.349170	0.980748	2.153457	23.969420
25	0.094629	0.000035	0.000697	2.365716	10.651660	0.078431	11.429820	0.981431	2.063959	23.152630
26	0.094629	0.000035	0.000697	2.460344	11.708840	0.086215	11.519140	0.982115	1.975185	22.345510
27	0.094629	0.000035	0.000697	2.554973	12.870950	0.094772	11.618130	0.982800	1.887194	21.548540
28	0.094629	0.000035	0.000697	2.649602	14.148400	0.104178	11.727930	0.983485	1.800049	20.762210
29	0.094629	0.000035	0.000697	2.744230	15.552640	0.114518	11.849820	0.984170	1.713821	19.987000
30	0.094629	0.000035	0.000697	2.838859	17.096240	0.125884	11.985280	0.984856	1.628581	19.223410
31	0.094629	0.000035	0.000697	2.933487	18.793060	0.138378	12.135960	0.985543	1.544411	18.471940
32	0.094629	0.000035	0.000697	3.028116	20.658280	0.152112	12.303790	0.986230	1.461394	17.733080
33	0.094629	0.000035	0.000697	3.122745	22.708620	0.167209	12.490950	0.986917	1.379621	17.007320
34	0.094629	0.000035	0.000697	3.217373	24.962470	0.183805	12.699970	0.987605	1.299188	16.295140
35	0.094629	0.000035	0.000697	3.312002	27.440000	0.202047	12.933780	0.988293	1.220198	15.597030
36	0.094629	0.000035	0.000697	3.406631	30.163440	0.222101	13.195760	0.988982	1.142760	14.913440

37	0.094629	0.000035	0.000697	3.501259	33.157180	0.244144	13.489880	0.989672	1.066985	14.244830
38	0.094629	0.000035	0.000697	3.595888	36.448040	0.268376	13.820750	0.990361	0.992993	13.591630
39	0.094629	0.000035	0.000697	3.690516	40.065530	0.295012	14.193830	0.991052	0.920909	12.954260
40	0.094629	0.000035	0.000697	3.785145	44.042060	0.324292	14.615570	0.991742	0.850858	12.333090
41	0.094629	0.000035	0.000697	3.879774	48.413260	0.356479	15.093640	0.992434	0.782973	11.728500
42	0.094629	0.000035	0.000697	3.974402	53.218300	0.391859	15.637230	0.993125	0.717386	11.140820
43	0.094629	0.000035	0.000697	4.069031	58.500250	0.430751	16.257380	0.993818	0.654232	10.570350
44	0.094629	0.000035	0.000697	4.163660	64.306430	0.473504	16.967490	0.994510	0.593644	10.017350
45	0.094629	0.000035	0.000697	4.258288	70.688870	0.520499	17.783920	0.995204	0.535752	9.482070
46	0.094629	0.000035	0.000697	4.352917	77.704780	0.572159	18.726780	0.995897	0.480682	8.964686
47	0.094629	0.000035	0.000697	4.447545	85.417030	0.628946	19.820990	0.996591	0.428551	8.465349
48	0.094629	0.000035	0.000697	4.542174	93.894710	0.691369	21.097710	0.997286	0.379467	7.984163
49	0.094629	0.000035	0.000697	4.636803	103.213800	0.759988	22.596240	0.997981	0.333525	7.521189
50	0.094629	0.000035	0.000697	4.731431	113.457800	0.835418	24.366590	0.998677	0.290800	7.076438
51	0.094629	0.000035	0.000697	4.826060	124.718600	0.918333	26.473080	0.999373	0.251352	6.649878
52	0.094629	0.000035	0.000697	4.920689	137.097000	1.009478	28.999350	1.000069	0.215211	6.241427
53	0.094629	0.000035	0.000697	5.015317	150.703900	1.109670	32.055370	1.000767	0.182387	5.850956
54	0.094629	0.000035	0.000697	5.109946	165.661400	1.219805	35.787550	1.001464	0.152854	5.478291
55	0.094629	0.000035	0.000697	5.204574	182.103400	1.340871	40.393400	1.002162	0.126559	5.123211
56	0.094629	0.000035	0.000697	5.299203	200.177200	1.473953	46.143150	1.002861	0.103413	4.785450
57	0.094629	0.000035	0.000697	5.393832	220.044900	1.620244	53.412200	1.003560	0.083293	4.464702
58	0.094629	0.000035	0.000697	5.488460	241.884500	1.781054	62.730600	1.004259	0.066044	4.160619
59	0.094629	0.000035	0.000697	5.583089	265.891700	1.957825	74.860010	1.004959	0.051479	3.872817
60	0.094629	0.000035	0.000697	5.677718	292.281600	2.152140	90.915910	1.005660	0.039384	3.600876
61	0.094629	0.000035	0.000697	5.772346	321.290700	2.365741	112.565600	1.006361	0.029522	3.344346
62	0.094629	0.000035	0.000697	5.866975	353.178900	2.600542	142.356900	1.007062	0.021643	3.102749
63	0.094629	0.000035	0.000697	5.961603	388.232100	2.858647	184.277400	1.007764	0.015484	2.875583
64	0.094629	0.000035	0.000697	6.056232	426.764400	3.142369	244.732200	1.008466	0.010787	2.662326
65	0.094629	0.000035	0.000697	6.150861	469.121000	3.454251	334.302600	1.009169	0.007299	2.462439
66	0.094629	0.000035	0.000697	6.245489	515.681500	3.797088	471.011700	1.009873	0.004784	2.275372
67	0.094629	0.000035	0.000697	6.340118	566.863200	4.173951	686.595900	1.010577	0.003027	2.100565
68	0.094629	0.000035	0.000697	6.434747	623.124700	4.588218	1038.999000	1.011281	0.001844	1.937453
69	0.094629	0.000035	0.000697	6.529375	684.970100	5.043601	1638.270000	1.011986	0.001077	1.785471
70	0.094629	0.000035	0.000697	6.624004	752.953800	5.544181	2702.618000	1.012691	0.000601	1.644053
71	0.094629	0.000035	0.000697	6.718632	827.684900	6.094444	4685.552000	1.013397	0.000319	1.512643
72	0.094629	0.000035	0.000697	6.813261	909.833100	6.699321	8579.372000	1.014103	0.000160	1.390689
73	0.094629	0.000035	0.000697	6.907890	1000.134000	7.364232	16681.030000	1.014810	0.000076	1.277650
74	0.094629	0.000035	0.000697	7.002518	1099.398000	8.095136	34645.780000	1.015518	0.000033	1.173001
75	0.094629	0.000035	0.000697	7.097147	1208.514000	8.898583	77371.820000	1.016225	0.000014	1.076228
76	0.094629	0.000035	0.000697	7.191776	1328.460000	9.781772	187131.500000	1.016934	0.000005	0.986836
77	0.094629	0.000035	0.000697	7.286404	1460.310000	10.752620	494060.500000	1.017643	0.000002	0.904349
78	0.094629	0.000035	0.000697	7.381033	1605.247000	11.819820	1436351.000000	1.018352	0.000001	0.828307
79	0.094629	0.000035	0.000697	7.475662	1764.569000	12.992950	4642392.000000	1.019062	0.000000	0.758273
80	0.094629	0.000035	0.000697	7.570290	1939.703000	14.282500	16857348.000000	1.019772	0.000000	0.693829
81	0.094629	0.000035	0.000697	7.664919	2132.220000	15.700050	69569953.000000	1.020483	0.000000	0.634578
82	0.094629	0.000035	0.000697	7.759547	2343.844000	17.258290	3.30E+08	1.021194	0.000000	0.580145
83	0.094629	0.000035	0.000697	7.854176	2576.471000	18.971180	1.83E+09	1.021906	0.000000	0.530175

84	0.094629	0.000035	0.000697	7.948805	2832.187000	20.854080	1.20E+10	1.022618	0.000000	0.484333
85	0.094629	0.000035	0.000697	8.043433	3113.284000	22.923860	9.54E+10	1.023331	0.000000	0.442305
86	0.094629	0.000035	0.000697	8.138062	3422.279000	25.199070	9.29E+11	1.024044	0.000000	0.403799
87	0.094629	0.000035	0.000697	8.232691	3761.942000	27.700090	1.13E+13	1.024758	0.000000	0.368537
88	0.094629	0.000035	0.000697	8.327319	4135.317000	30.449340	1.77E+14	1.025472	0.000000	0.336265
89	0.094629	0.000035	0.000697	8.421948	4545.749000	33.471450	3.63E+15	1.026187	0.000000	0.306743
90	0.094629	0.000035	0.000697	8.516576	4996.917000	36.793510	1.01E+17	1.026902	0.000000	0.279748
91	0.094629	0.000035	0.000697	8.611205	5492.864000	40.445280	3.88E+18	1.027618	0.000000	0.255075
92	0.094629	0.000035	0.000697	8.705834	6038.033000	44.459500	2.15E+20	1.028334	0.000000	0.232534
93	0.094629	0.000035	0.000697	8.800462	6637.312000	48.872130	1.77E+22	1.029051	0.000000	0.211946
94	0.094629	0.000035	0.000697	8.895091	7296.068000	53.722720	2.27E+24	1.029768	0.000000	0.193149
95	0.094629	0.000035	0.000697	8.989720	8020.207000	59.054720	4.69E+26	1.030486	0.000000	0.175993
96	0.094629	0.000035	0.000697	9.084348	8816.217000	64.915940	1.65E+29	1.031204	0.000000	0.160338
97	0.094629	0.000035	0.000697	9.178977	9691.232000	71.358880	1.03E+32	1.031923	0.000000	0.146057
98	0.094629	0.000035	0.000697	9.273605	10653.090000	78.441290	1.23E+35	1.032642	0.000000	0.133033
99	0.094629	0.000035	0.000697	9.368234	11710.420000	86.226630	2.96E+38	1.033362	0.000000	0.121157
100	0.094629	0.000035	0.000697	9.462863	12872.680000	94.784670	1.54E+42	1.034083	0.000000	0.110330
101	0.094629	0.000035	0.000697	9.557491	14150.300000	104.192100	1.88E+46	1.034803	0.000000	0.100462
102	0.094629	0.000035	0.000697	9.652120	15554.730000	114.533200	5.82E+50	1.035525	0.000000	0.091469
103	0.094629	0.000035	0.000697	9.746749	17098.540000	125.900700	5.03E+55	1.036246	0.000000	0.083275
104	0.094629	0.000035	0.000697	9.841377	18795.580000	138.396400	1.35E+61	1.036969	0.000000	0.075810
105	0.094629	0.000035	0.000697	9.936006	20661.050000	152.132300	1.24E+67	1.037691	0.000000	0.069009
106	0.094629	0.000035	0.000697	10.030630	22711.680000	167.231600	4.48E+73	1.038415	0.000000	0.062815
107	0.094629	0.000035	0.000697	10.125260	24965.820000	183.829400	7.25E+80	1.039138	0.000000	0.057174
108	0.094629	0.000035	0.000697	10.219890	27443.690000	202.074600	6.08E+88	1.039863	0.000000	0.052038
109	0.094629	0.000035	0.000697	10.314520	30167.500000	222.130600	3.12E+97	1.040588	0.000000	0.047360
110	0.094629	0.000035	0.000697	10.409150	33161.640000	244.177100	1.17E+107	1.041313	0.000000	0.043101
111	0.094629	0.000035	0.000697	10.503780	36452.950000	268.411900	3.92E+117	1.042039	0.000000	0.039224
112	0.094629	0.000035	0.000697	10.598410	40070.920000	295.051900	1.46E+129	1.042765	0.000000	0.035695
113	0.094629	0.000035	0.000697	10.693030	44047.980000	324.335900	7.61E+141	1.043492	0.000000	0.032482
114	0.094629	0.000035	0.000697	10.787660	48419.770000	356.526500	7.27E+155	1.044219	0.000000	0.029557
115	0.094629	0.000035	0.000697	10.882290	53225.460000	391.911900	1.69E+171	1.044947	0.000000	0.026895
116	0.094629	0.000035	0.000697	10.976920	58508.110000	430.809400	1.32E+188	1.045675	0.000000	0.024473
117	0.094629	0.000035	0.000697	11.071550	64315.070000	473.567400	4.92E+206	1.046404	0.000000	0.022268
118	0.094629	0.000035	0.000697	11.166180	70698.380000	520.569300	1.27E+227	1.047134	0.000000	0.020261
119	0.094629	0.000035	0.000697	11.260810	77715.230000	572.236100	3.49E+249	1.047863	0.000000	0.018435

The females' parameters are

$\rho = 0.0002462199082$; $G = 0.00001878394898$ and $H = 1.102973884$

Table 2: Female life expectancy

Age	2	3	4	5	6	7	8	9	10	11
0	0.098010	0.000246	0.000019	0.000000	1.000000	0.000192	10.204990	0.998361	7.989635	81.400490
1	0.098010	0.000246	0.000019	0.098010	1.102974	0.000211	10.205190	0.998380	7.891484	80.403610
2	0.098010	0.000246	0.000019	0.196020	1.216551	0.000233	10.205410	0.998398	7.793338	79.406850
3	0.098010	0.000246	0.000019	0.294030	1.341824	0.000257	10.205660	0.998417	7.695196	78.410220

4	0.098010	0.000246	0.000019	0.392040	1.479997	0.000284	10.205930	0.998436	7.597058	77.413750
5	0.098010	0.000246	0.000019	0.490050	1.632398	0.000313	10.206230	0.998455	7.498924	76.417440
6	0.098010	0.000246	0.000019	0.588060	1.800493	0.000345	10.206560	0.998473	7.400795	75.421310
7	0.098010	0.000246	0.000019	0.686070	1.985896	0.000381	10.206920	0.998492	7.302672	74.425380
8	0.098010	0.000246	0.000019	0.784081	2.190392	0.000420	10.207320	0.998511	7.204554	73.429670
9	0.098010	0.000246	0.000019	0.882091	2.415945	0.000463	10.207760	0.998530	7.106442	72.434190
10	0.098010	0.000246	0.000019	0.980101	2.664724	0.000511	10.208250	0.998548	7.008336	71.438970
11	0.098010	0.000246	0.000019	1.078111	2.939121	0.000563	10.208780	0.998567	6.910237	70.444030
12	0.098010	0.000246	0.000019	1.176121	3.241774	0.000621	10.209380	0.998586	6.812146	69.449400
13	0.098010	0.000246	0.000019	1.274131	3.575592	0.000685	10.210030	0.998605	6.714062	68.455110
14	0.098010	0.000246	0.000019	1.372141	3.943785	0.000756	10.210750	0.998623	6.615986	67.461180
15	0.098010	0.000246	0.000019	1.470151	4.349892	0.000834	10.211540	0.998642	6.517920	66.467650
16	0.098010	0.000246	0.000019	1.568161	4.797817	0.000920	10.212420	0.998661	6.419864	65.474550
17	0.098010	0.000246	0.000019	1.666171	5.291867	0.001014	10.213390	0.998680	6.321818	64.481920
18	0.098010	0.000246	0.000019	1.764181	5.836791	0.001119	10.214450	0.998698	6.223784	63.489810
19	0.098010	0.000246	0.000019	1.862191	6.437828	0.001234	10.215630	0.998717	6.125762	62.498250
20	0.098010	0.000246	0.000019	1.960201	7.100756	0.001361	10.216930	0.998736	6.027754	61.507290
21	0.098010	0.000246	0.000019	2.058211	7.831948	0.001501	10.218360	0.998755	5.929761	60.516980
22	0.098010	0.000246	0.000019	2.156221	8.638435	0.001656	10.219940	0.998773	5.831784	59.527390
23	0.098010	0.000246	0.000019	2.254231	9.527968	0.001826	10.221680	0.998792	5.733825	58.538560
24	0.098010	0.000246	0.000019	2.352242	10.509100	0.002014	10.223600	0.998811	5.635886	57.550560
25	0.098010	0.000246	0.000019	2.450252	11.591260	0.002222	10.225730	0.998830	5.537967	56.563460
26	0.098010	0.000246	0.000019	2.548262	12.784860	0.002450	10.228060	0.998849	5.440072	55.577340
27	0.098010	0.000246	0.000019	2.646272	14.101370	0.002703	10.230650	0.998867	5.342202	54.592270
28	0.098010	0.000246	0.000019	2.744282	15.553440	0.002981	10.233490	0.998886	5.244360	53.608340
29	0.098010	0.000246	0.000019	2.842292	17.155040	0.003288	10.236630	0.998905	5.146549	52.625640
30	0.098010	0.000246	0.000019	2.940302	18.921560	0.003626	10.240100	0.998924	5.048770	51.644270
31	0.098010	0.000246	0.000019	3.038312	20.869980	0.004000	10.243930	0.998942	4.951029	50.664330
32	0.098010	0.000246	0.000019	3.136322	23.019050	0.004412	10.248150	0.998961	4.853327	49.685930
33	0.098010	0.000246	0.000019	3.234332	25.389410	0.004866	10.252800	0.998980	4.755670	48.709200
34	0.098010	0.000246	0.000019	3.332342	28.003850	0.005367	10.257940	0.998999	4.658061	47.734270
35	0.098010	0.000246	0.000019	3.430352	30.887520	0.005920	10.263610	0.999017	4.560505	46.761260
36	0.098010	0.000246	0.000019	3.528362	34.068130	0.006529	10.269870	0.999036	4.463008	45.790340
37	0.098010	0.000246	0.000019	3.626372	37.576250	0.007202	10.276780	0.999055	4.365575	44.821640
38	0.098010	0.000246	0.000019	3.724382	41.445630	0.007943	10.284400	0.999074	4.268213	43.855350
39	0.098010	0.000246	0.000019	3.822392	45.713440	0.008761	10.292820	0.999092	4.170928	42.891640
40	0.098010	0.000246	0.000019	3.920403	50.420740	0.009663	10.302110	0.999111	4.073729	41.930690
41	0.098010	0.000246	4000.000019	4.018413	55.612750	0.010658	10.312360	0.999130	3.976623	40.972710
42	0.098010	0.000246	0.000019	4.116423	61.339420	0.011756	10.323690	0.999149	3.879621	40.017900
43	0.098010	0.000246	0.000019	4.214433	67.655770	0.012966	10.336190	0.999168	3.782732	39.066500
44	0.098010	0.000246	0.000019	4.312443	74.622550	0.014302	10.350000	0.999186	3.685969	38.118750
45	0.098010	0.000246	0.000019	4.410453	82.306730	0.015774	10.365260	0.999205	3.589343	37.174880
46	0.098010	0.000246	0.000019	4.508463	90.782170	0.017399	10.382110	0.999224	3.492868	36.235180
47	0.098010	0.000246	0.000019	4.606473	100.130400	0.019190	10.400720	0.999243	3.396559	35.299920
48	0.098010	0.000246	0.000019	4.704483	110.441200	0.021166	10.421300	0.999261	3.300434	34.369400
49	0.098010	0.000246	0.000019	4.802493	121.813700	0.023346	10.444040	0.999280	3.204509	33.443920
50	0.098010	0.000246	0.000019	4.900503	134.357400	0.025750	10.469170	0.999299	3.108806	32.523810

51	0.098010	0.000246	0.000019	4.998513	148.192700	0.028402	10.496970	0.999318	3.013346	31.609420
52	0.098010	0.000246	0.000019	5.096523	163.452600	0.031326	10.527720	0.999336	2.918153	30.701100
53	0.098010	0.000246	0.000019	5.194533	180.284000	0.034552	10.561730	0.999355	2.823254	29.799220
54	0.098010	0.000246	0.000019	5.292543	198.848500	0.038110	10.599380	0.999374	2.728678	28.904180
55	0.098010	0.000246	0.000019	5.390553	219.324700	0.042034	10.641050	0.999393	2.634457	28.016370
56	0.098010	0.000246	0.000019	5.488564	241.909500	0.046363	10.687210	0.999412	2.540625	27.136220
57	0.098010	0.000246	0.000019	5.586574	266.819800	0.051137	10.738360	0.999430	2.447220	26.264150
58	0.098010	0.000246	0.000019	5.684584	294.295300	0.056403	10.795050	0.999449	2.354285	25.400620
59	0.098010	0.000246	0.000019	5.782594	324.600000	0.062211	10.857930	0.999468	2.261864	24.546090
60	0.098010	0.000246	0.000019	5.880604	358.025300	0.068617	10.927710	0.999487	2.170007	23.701030
61	0.098010	0.000246	0.000019	5.978614	394.892600	0.075682	11.005200	0.999505	2.078768	22.865930
62	0.098010	0.000246	0.000019	6.076624	435.556200	0.083476	11.091300	0.999524	1.988205	22.041280
63	0.098010	0.000246	0.000019	6.174634	480.407100	0.092072	11.187050	0.999543	1.898382	21.227590
64	0.098010	0.000246	0.000019	6.272644	529.876500	0.101553	11.293620	0.999562	1.809369	20.425360
65	0.098010	0.000246	0.000019	6.370654	584.440000	0.112010	11.412340	0.999581	1.721239	19.635120
66	0.098010	0.000246	0.000019	6.468664	644.622000	0.123544	11.544730	0.999599	1.634074	18.857390
67	0.098010	0.000246	0.000019	6.566674	711.001300	0.136266	11.692540	0.999618	1.547961	18.092680
68	0.098010	0.000246	0.000019	6.664684	784.215800	0.150298	11.857760	0.999637	1.462992	17.341510
69	0.098010	0.000246	0.000019	6.762694	864.969600	0.165774	12.042710	0.999656	1.379268	16.604400
70	0.098010	0.000246	0.000019	6.860704	954.038800	0.182845	12.250050	0.999674	1.296896	15.881860
71	0.098010	0.000246	0.000019	6.958714	1052.280000	0.201673	12.482880	0.999693	1.215989	15.174390
72	0.098010	0.000246	0.000019	7.056725	1160.637000	0.222440	12.744820	0.999712	1.136669	14.482470
73	0.098010	0.000246	0.000019	7.154735	1280.153000	0.245345	13.040120	0.999731	1.059061	13.806560
74	0.098010	0.000246	0.000019	7.252745	1411.975000	0.270610	13.373760	0.999750	0.983300	13.147120
75	0.098010	0.000246	0.000019	7.350755	1557.371000	0.298475	13.751670	0.999768	0.909524	12.504580
76	0.098010	0.000246	0.000019	7.448765	1717.740000	0.329210	14.180890	0.999787	0.837878	11.879330
77	0.098010	0.000246	0.000019	7.546775	1894.622000	0.363111	14.669870	0.999806	0.768509	11.271740
78	0.098010	0.000246	0.000019	7.644785	2089.719000	0.400501	15.228770	0.999825	0.701568	10.682150
79	0.098010	0.000246	0.000019	7.742795	2304.906000	0.441743	15.869960	0.999843	0.637206	10.110850
80	0.098010	0.000246	0.000019	7.840805	2542.251000	0.487231	16.608520	0.999862	0.575574	9.558110
81	0.098010	0.000246	0.000019	7.938815	2804.036000	0.537403	17.463060	0.999881	0.516818	9.024142
82	0.098010	0.000246	0.000019	8.036825	3092.778000	0.592741	18.456680	0.999900	0.461078	8.509117
83	0.098010	0.000246	0.000019	8.134835	3411.254000	0.653778	19.618310	0.999919	0.408487	8.013162
84	0.098010	0.000246	0.000019	8.232845	3762.524000	0.721100	20.984520	0.999937	0.359161	7.536351
85	0.098010	0.000246	0.000019	8.330855	4149.966000	0.795354	22.602030	0.999956	0.313203	7.078709
86	0.098010	0.000246	0.000019	8.428865	4577.304000	0.877255	24.531070	0.999975	0.270692	6.640209
87	0.098010	0.000246	0.000019	8.526875	5048.646000	0.967590	26.850240	0.999994	0.231685	6.220772
88	0.098010	0.000246	0.000019	8.624886	5568.525000	1.067226	29.663320	1.000012	0.196208	5.820265
89	0.098010	0.000246	0.000019	8.722896	6141.938000	1.177122	33.109080	1.000031	0.164255	5.438507
90	0.098010	0.000246	0.000019	8.820906	6774.397000	1.298335	37.375690	1.000050	0.135784	5.075263
91	0.098010	0.000246	0.000019	8.918916	7471.983000	1.432030	42.722050	1.000069	0.110714	4.730249
92	0.098010	0.000246	0.000019	9.016926	8241.402000	1.579492	49.510110	1.000088	0.088926	4.403135
93	0.098010	0.000246	0.000019	9.114936	9090.051000	1.742138	58.254610	1.000106	0.070262	4.093546
94	0.098010	0.000246	0.000019	9.212946	10026.090000	1.921533	69.701230	1.000125	0.054527	3.801066
95	0.098010	0.000246	0.000019	9.310956	11058.510000	2.119400	84.951930	1.000144	0.041491	3.525237
96	0.098010	0.000246	0.000019	9.408966	12197.250000	2.337643	105.670800	1.000163	0.030898	3.265572
97	0.098010	0.000246	0.000019	9.506976	13453.250000	2.578360	134.430100	1.000182	0.022473	3.021549

98	0.098010	0.000246	0.000019	9.604986	14838.580000	2.843863	175.308700	1.000200	0.015927	2.792622
99	0.098010	0.000246	0.000019	9.702996	16366.570000	3.136707	234.954500	1.000219	0.010971	2.578220
100	0.098010	0.000246	0.000019	9.801006	18051.900000	3.459706	324.534200	1.000238	0.007325	2.377759
101	0.098010	0.000246	0.000019	9.899016	19910.780000	3.815965	463.427700	1.000257	0.004726	2.190638
102	0.098010	0.000246	0.000019	9.997026	21961.070000	4.208910	686.492300	1.000275	0.002936	2.016248
103	0.098010	0.000246	0.000019	10.095040	24222.480000	4.642318	1058.918000	1.000294	0.001750	1.853976
104	0.098010	0.000246	0.000019	10.193050	26716.760000	5.120355	1707.935000	1.000313	0.000997	1.703208
105	0.098010	0.000246	0.000019	10.291060	29467.890000	5.647618	2893.735000	1.000332	0.000540	1.563334
106	0.098010	0.000246	0.000019	10.389070	32502.320000	6.229175	5176.376000	1.000351	0.000277	1.433751
107	0.098010	0.000246	0.000019	10.487080	35849.210000	6.870618	9831.067000	1.000369	0.000134	1.313865
108	0.098010	0.000246	0.000019	10.585090	39540.740000	7.578112	19946.260000	1.000388	0.000060	1.203095
109	0.098010	0.000246	0.000019	10.683100	43612.400000	8.358459	43527.340000	1.000407	0.000025	1.100878
110	0.098010	0.000246	0.000019	10.781110	48103.340000	9.219162	102934.400000	1.000426	0.000010	1.006664
111	0.098010	0.000246	0.000019	10.879120	53056.730000	10.168500	265981.100000	1.000445	0.000003	0.919928
112	0.098010	0.000246	0.000019	10.977130	58520.190000	11.215580	757872.200000	1.000463	0.000001	0.840161
113	0.098010	0.000246	0.000019	11.075140	64546.240000	12.370500	2405293.000000	1.000482	0.000000	0.766880
114	0.098010	0.000246	0.000019	11.173150	71192.810000	13.644340	8597831.000000	1.000501	0.000000	0.699622
115	0.098010	0.000246	0.000019	11.271160	78523.810000	15.049350	35041049.000000	1.000520	0.000000	0.637949
116	0.098010	0.000246	0.000019	11.369170	86609.720000	16.599040	165000000.000000	1.000539	0.000000	0.581446
117	0.098010	0.000246	0.000019	11.467180	95528.250000	18.308300	912000000.000000	1.000557	0.000000	0.529720
118	0.098010	0.000246	0.000019	11.565190	105365.200000	20.193580	6010000000.000000	1.000576	0.000000	0.482405
119	0.098010	0.000246	0.000019	11.663200	116215.000000	22.272990	48100000000.000000	1.000595	0.000000	0.439154

Discussion of Results

It is evident from the Tables that average life expectancy reduces as age progressively advances. Specifically, from Tables $1-2$, the individual $\bar{e}_x(male)$ life expectancies is much lower than the $\bar{e}_x(female)$ life expectancies. The reason could be attributed to the level of total hazards experienced by men. In Table 1, within the interval $75 \le x \le 119$, life expectancy sharply decreases. However in Table 2, it is observed that within the interval $111 \le x \le 119$ for the female, there is a marked decrease in life expectancies. In the study of longevity, the intensity of death in health care systems is often anchored around the modal age at death x_M and this accounts for why the modal age at death has been suggested as equivalent to complete life expectancy. The real number x_M will occur where the survival function l_x in flexes such that the mean of the distribution of death (life expectancy) will be much dependent on the left tail of mortality at young ages. For example for males in Table 1, $x_M = 77.7$ years and for females $x_W = 113$ years.

$$x_M = \frac{1}{\log_e H}\log_e\left[\frac{-\{2\rho G - G(\log_e H)\} + \sqrt{(2(\log_e H)G - G\beta)^2 - 4G^2\rho^2}}{2G^2}\right] \qquad (146)$$

Life expectancy oftentimes quoted at birth in a $GM(1,2)$ based mortality model defines the expected present value of the underlying death density or distribution and is equal to the total expected lifespan. Specifically, a clear advantage of the $GM(1,2)$ model used here to mortality indicators is that it could be applied to explain life expectancy patterns for historical population profile which could be employed as inputs for mortality projections. Consequently, a $GM(1,2)$ based life expectancy values will serve as useful analytical devices to the high-quality human mortality databases applying complex numerical processes in estimating life expectancies. The computed complete life expectancies can be used to estimate life table exposures especially the person-years lived. This paper therefore contributes to this field by providing analytical technique of calculating the life expectancies under the $GM(1,2)$

instantaneous intensity describing the pattern of human mortality. As shown in Tables $1-2$, life expectancy can be computed at any defined age where the survivor's age and his life expectancy imply the total expected lifespan. In Tables $1-2$, the life expectancies at birth are $\bar{e}_0(male)=45.4$ and $\bar{e}_0(female)=81.4$ years respectively which means that men roughly live on average half as half many years as women. The computations have implications on period life expectancy which is usually captured as summary indicator of mortality at a specified time and which does not permit any modification for changes to mortality beyond the year under reference. In our computation, for example $\bar{e}_{60}(2020)$ will use instantaneous death μ_x at age 60 in 2020, use μ_x when (x) reaches 61 in 2020, use μ_x when (x) reaches 62 in 2020 and so on. This trend makes it an objective standard longevity measure over different populations and consequently $\bar{e}_x$ can be estimated directly from observed death probability in a specified year without accounting for any mortality projection technique for period life expectancy.

$$\bar{e}_x(\xi)=\frac{1}{2}+\sum_{u=1}^{\Omega-x-1} {}_u p_x(\xi)=f\left(\mu_0(\xi),\mu_1(\xi),\mu_2(\xi),\ldots\right) \tag{147}$$

$$\bar{e}_x(\xi)=\frac{1}{2}+\sum_{u=1}^{\Omega-x-1}\left[p_x(\xi)\times p_{x+1}(\xi)\times p_{x+2}(\xi)\times p_{x+3}(\xi)\times\ldots\times p_{x+u}(\xi)\right] \tag{148}$$

However, since longevity improves over the passage of time, period life expectancy can possibly and inadvertently underestimate the true complete life expectancy of a life. Consequently, this accounts for the reason why cohort life expectancy is computed factoring into accounts future improvements in longevity by using reduction factor function in mortality intensities for a life surviving age x after ξ years from base year. Further to the arguments above, the cohort life expectancy for an individual aged 60 in 2020 will use instantaneous death μ_x at age 60 in 2020 still surviving 2020, μ_x at 61 in 2021 surviving 1 year after 2020, μ_x at 62 in 2022 surviving 2 years after 2020. Such computations need projections of future probabilities $p_{x+u}(\xi+u)$ for future years $\xi+1;\xi+2;\xi+3;\ldots$ for cohort life expectancy.

$$\bar{e}_x(\xi)=\frac{1}{2}+\sum_{u=1}^{\Omega-x-1} {}_u p_x(\xi)=f\left(\mu_0(\xi+1),\mu_1(\xi+2),\mu_2(\xi+3),\ldots\right) \tag{149}$$

$$\bar{e}_x(\xi)=\frac{1}{2}+\sum_{u=1}^{\Omega-x-1}\left[p_x(\xi)\times p_{x+1}(\xi+1)\times p_{x+2}(\xi+2)\times p_{x+3}(\xi+3)\times\ldots\times p_{x+u}(\xi+u)\right] \tag{150}$$

Since the parameters of the underlying mortality laws for cohort life expectancy are time dependent, estimating cohort life expectancy is somewhat complex because more cohorts can only be observed over a restricted number of years making the estimation of time varying parameters unstable. Moreover, mortality projection suffers basis risk problem arising because the scheme holder's mortality intensities with the pace of their evolution markedly diverges from the general population which are employed in modelling mortality projections. This divergence falls in line with the observations in Committee of Mortality Investigation (CMI) $GM(m,n)$ as a result of insurance portfolio-specific selection effect. When there is an improvement in mortality relative to the expectation of life, the life office's liability from life insurance schemes decline since death benefits payments are then postponed relative to the initial expectations. Simultaneously, the life office will incur losses on annuity schemes relative to the initial policy design since annuity payments disbursed to long-lived annuity holders will increase. However, this inverse consequential effect in mortality improvement connected with life insurance schemes will most likely create an actuarial hedging strategy against longevity risk. We identify two basic consequences of the results obtained on life insurance underwriting: (i) the average value of the remaining life time for (x) derived by adding death probabilities within a defined time interval until age 119 through the distribution of collective probability of death for the insured population is expressed as under guaranteed sum assured as death benefit.

$$ {}_{0|119-x}q_x = P\left(0 < T_x < 119 - x\right) = \sum_{k=0}^{119-x} {}_{k|1}q_x = \sum_{k=0}^{119-x} {}_1q_{x+k} \prod_{r=0}^{k-1}\left({}_1p_{x+r}\right) \qquad (151) $$

The expected value of death benefit B in respect of whole life insurance purchased by (x) is given by $\sum_{k=0}^{119-x} \frac{1}{(1+i)^{k+1}}\left[{}_1q_{x+k}\prod_{r=0}^{k-1}\left({}_1p_{x+r}\right)\right]$ and (ii) Assuming $\Omega = 119$ is the limiting age and suppose regular payments of life annuity commence at age α, then the expected remaining life time with regular payments for the annuity contracts purchased by (x) is expressed as

$$ P\left(\alpha - x < T_x < 119 - x\right) = \left({}_{\alpha-x}p_x\right)\sum_{k=0}^{119-\alpha}\left({}_kp_\alpha\right) = \sum_{k=0}^{119-\alpha}\left({}_{\alpha-x+k}p_x\right) = \sum_{k=0}^{119-\alpha}\left(\prod_{r=0}^{\alpha-x+k-1}\left({}_1p_{x+r}\right)\right) \qquad (152) $$

Assuming regular annuity payments are made at the beginning of each year and commences at age 70 till age 119 for the whole annuity due, then the net single premium for the whole life annuity due of 1 unit with entry age y where $y \le 70$ is expressed as

$$ {}_{70-y|}\ddot{a}_y = \sum_{k=0}^{49}\left({}_{70-y+k}p_y\right) \times e^{\delta(y-k-70)} \qquad (153) $$

Conclusion

Using Gradshteyn and Ryzhik's integral to estimate continuous life annuities has significant implications from the mathematical and practical standpoint. In the context of continuous life expectancies and annuities, the integral function is typically applied to evaluate integrals that arise from the mathematical modeling of such life expectancies. The ability to use specific Gradshteyn and Ryzhik integrals allows for a more nuanced financial model that incorporates continuous compounding and continuous time survival functions. This improves the function's ability to capture life underwriting conditions especially for long-term financial products such as life insurance and annuities where cash flows occur continuously rather than at discrete intervals. In estimating the present value of a continuous life annuity, it is required to compute integrals of the form $PV_x = \int_0^{\Omega-x} e^{-\delta t} M(t)\,dt$ where $M(t)$ is a mortality function. The Gradshteyn and Ryzhik provides frameworks for these kind of benefit integrals that assist actuaries to compute the present value more effectively. This is crucial for both pricing the annuity, expectancies and determining appropriate reserves since life annuities and life expectancies require incorporating survival probability distributions over time. The distributions usually involve complex mortality models from Makeham's law that are often difficult to integrate directly. The use of Gradshteyn and Ryzhik simplifies the integration process when such functions appeared in the annuity expression ensuring that mortality assumptions are accurately factored into the annuity pricing. Since certain types of mortality models or mortality rate structures may not have simple closed-form solutions, the Gradshteyn and Ryzhik allows a broader set of assumptions in life annuity models. With access to the integrals in Gradshteyn and Ryzhik, more complex and flexible assumptions can be incorporated into the life expectancy estimation, improving the robustness of the financial model. Instead of numerically approximating complex integrals, the mortality expression from Gradshteyn and Ryzhik provides closed-form expressions, making the process more efficient. However, the integrals often derived under idealized life insurance conditions can introduce additional complexities such as irregular mortality assumptions or varying interest rates.

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