*Lafia Journal of Scientific & Industrial Research* (LJSIR), Vol. 2(2), 2024
p-ISSN: 3026 – 9288 e-ISSN: 3027 – 1800 pages: 108 – 122
https://lafiascijournals.org.ng/index.php/ljsir/index ⇒ Published by the Faculty of Science,
Federal University of Lafia, Nasarawa State, Nigeria

# Generating Mortality Rate Intensity for Life Insurance Applications through Novel Method of Successive Differencing Under the Parsimonious Generalised Makeham's Framework

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## Abstract

Developing and implementing age dependent mortality rates in functional forms have presented critical modelling problems for life insurance and annuity firms; thereby creating serious research gaps. A recurring problem with the existing life tables is that their underlying generating functions are not capable of showing any evidence that mortality at 10 declines. This observed irregularity in the age pattern of mortality is the prime motivation in the search for analytical functions which fully capture the observed variations of mortality with age. To fill the gaps identified, the study aims to develop specific life table function under the Generalised Makeham's framework. The objectives of this study are to (i) compute the mortality rate intensities $\mu_x$ (ii) compute the curve of deaths densities $l_x\mu_x$ (iii) compute the probability of deaths $q_x$ and then compare the common domain of definition of these measures. From the method of successive differencing employed to model *GM* (2,2), the male's ageing parameter value 1.102923606 falls within the globally accepted interval 1.08≤C≤1.12 for the *GM* (2,2). This method is superior to the method of maximum likelihood estimation which mostly violates the permissible interval of validity. Computational evidence from our analysis proves that under the Generalised Makeham's law, the mortality rate intensity declines at 10.

**Keywords:** Generalised Makeham, curve of deaths, permissible interval, successive differencing

**Article History**
***Submitted***
April 18, 2024
***Revised***
July 25, 2024
***First Published Online***
July 28, 2024

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doi.org/10.62050/ljsir2024.v2n2.338

## Introduction

Life insurance which payments are either contingent on death or survival concerns every individual especially the working population who are automatically covered while in service by reason of the government's social security program but who will possibly choose life annuity options at retirement. In actuarial context, it is essential to observe that life tables are used to measure the pattern of deaths, survival rates, life expectancy at varying ages in order to explain changes in the population.

In order to price life insurance and annuity products and ensure the solvency of insurance companies, estimates of future events such as deaths are functionally formulated using mortality models to generate life tables. For life offices, any potential deviation from mortality assumption employed in pricing at the inception of the contract will constitute clear threats to its underwriting performance.It has been observed that many life insurance providers in developing countries such as Nigeria depend on exotic life tables which are incorrectly applied to overcharge or undercharge their policyholders. If they undercharge, many claims are reported and life insurance products are underpriced leading to the risk of insurer's insolvency or if they overcharge, workers will suffer inadequate income streams at retirement.

Developing and implementing age dependent mortality rates in functional forms have presented critical modelling problems for life insurance and annuity firms; thereby creating serious research gaps. A recurring problem with the existing life tables is that their underlying generating functions are not capable of measuring perinatal mortality and there is no evidence that mortality at 10 declines. These observed irregularities in the age pattern of mortality are the prime motivation in the search for analytical functions that fully capture the observed variations of mortality with age. To fill the gaps identified, the study aims to develop specific life table function under the frameworks of Generalised Makeham's force of mortality.

The basic data input required for generating life tables are the age specific death rates enumerated from the information on deaths from vital statistics (using age and sex) and population $l_x$ by age and sex. In most developing countries such as Nigeria, these data do not exist because of dysfunctional vital registration system.

The most popular law of mortality stated in Gompertz's law [1] was developed to model old age mortality around 60 - 90 years and assumes that death rates increaseexponentially at old age.

Following observation by Debon *et al.* [2], the Generalized Makeham's mortality function in equation (1) is defined to measure mortality from infancy to old ages.

$$\mu_x = GM(m,n) = \sum_{k=1}^{m} \beta_k x^{k-1} + \exp \sum_{k=m+1}^{m+n} \beta_k x^{k-m-1} \quad (1)$$

where $m$ and $n$ are non-negative integers representing the orders of the polynomials and $\beta_k$ are mortality parameters. Because of the complexity involved in evaluating the parameters as both orders increase, $m$ and $n$ are constrained inpractice as $m + n < 5$ . This equation has implicationsfor $m$ and $n$ . When $m = 0$ and $n = 0$ , we obtain thefollowing equations;

$$\mu_x = GM(0,n) = \sum_{k=1}^{0} \beta_k x^{k-1} + \exp \sum_{k=1}^{n} \beta_k x^{k-1} = \beta_1 + \frac{\beta_0}{x} + \exp \sum_{k=1}^{n} \beta_k x^{k-1} \quad (2)$$

$$\mu_x = GM(m,0) = \sum_{k=1}^{m} \beta_k x^{k-1} + \exp \sum_{k=m+1}^{m} \beta_k x^{k-m-1} = \sum_{k=1}^{m} \beta_k x^{k-1} + \exp\left( (\beta_{m+1}) + \left( \frac{\beta_m}{x} \right) \right) \quad (3)$$

Consequently, mortality at zero $\mu_0$ is inadmissible. Therefore, in order to absorb the Gompertz's assumption and fall in line with actuarial practice, the undefined terms $\frac{\beta_0}{x}$ and $\frac{\beta_m}{x}$ at age zero are ignored and the equations are modified as follows;

$$\mu_x = GM(m,n) = \begin{cases} \sum_{k=1}^{m} \beta_k x^{k-1} + \exp \sum_{k=m+1}^{m+n} \beta_k x^{k-m-1} & m \geq 1, n \geq 1 \\ \exp \sum_{k=1}^{n} \beta_k x^{k-1} & m = 0 \\ \sum_{k=1}^{m} \beta_k x^{k-1} & n = 0 \end{cases} \quad (4)$$

If $m = 0, n = 2$, we have the Gompertz's law $\mu_x = GM(0,2) = BC^x$; $60 \leq x \leq 90$ where $B$ is the initial mortality, $C$ is the ageing parameter and $x$ is the age. This mortality law is not capable of capturing infant mortality rates and mortality due to accidents in young adulthood (accidental hump). This is because exponential mortality growth rate is not expected before sexual maturity.

The relatively few parameters in [1] makes it inflexible consequently, the author in [3] assumes a constant additive age-independent constant parameter that accounts for the accidental hump mortality. If $m = 1, n = 2$ in equation(4) the Makeham's mortality function becomes

$$\mu_x = GM(1,2) = A + BC^x; \quad 20 \leq x \leq 90 \quad (4a)$$

where $A$ defines the background mortality independent of age [4]. The assumption of constant mortality rate is not always true since we all have different mortality exposures due to different life styles, nutrition, job and environment.

To capture mortality at low ages and extreme ages from this law, mortality rates are then extrapolated because of insufficient mortality experience. Since there is an approved range of validity where the mortality function is well behaved, best mortality estimates outside the intended range may not be obtained by extrapolation. In order to avoid the need to extrapolate, we can set $m = 2, n = 2$ in (4) to obtain the Generalized Makeham's law $\mu_x = GM(2,2) = A + Hx + BC^x$ where $H \in \mathbf{R}$ is a new parameter which explains changes in the background mortality. It is natural for mortality to decline at ten to explain changes in mortality between conception and death. The Heligman-Pollard mortality law of the form $\mu_x = \sum_{k=1}^{m} a_k \exp\left[ -b_k \left( f_k(x) - c_k \right)^{\alpha_k} \right]$ was developed to decompose mortality according to three stages of life where the constants $\{a_k, b_k, c_k, \alpha_k\}$ are mortality parameters and $f_k(x) = \ln x$ or $f_k(x) = x$ [2]. However, the major problem currently is that, the generating survival function, the probability of survival function and the hazard rate functions have not been developed as a result of the emerging analytical intractability of the force of mortality function and this accounts for the reason why its associated life insurance monetary functions have not been developed till date.

Since mortality intensity evolves in continuous time, a basic functional representation of mortality risks is the severity to die function which defines the aggregate death severity that a life is exposed to over time. Consequently, the probability that a life survives to an age is the exponentiated negative severity to die function within an arbitrarily closed time interval.

A sound knowledge of mortality intensity could be applied to measure: (i) the severity to die function $\int_{\mathbf{R}^+} \mu_{x+t} dt$ (ii) the survival probability function $_t p_x$ (iii) the curve of death $l_x \mu_x$ of an insured life all applicable to carry out actuarial valuations in life insurance schemes. Thus all life contingent events represent direct application of numerical techniques in computing life table functions to ensure cost effective approximations.

Actuaries are required to conduct precise mortality estimations using the appropriate numerical technique since valuation of life insurance schemes depend heavily on both mortality rate and interest rate intensities when performing actuarial valuations on life insurance and pension schemes.

The inability to conduct accurate mortality results will expose life insurance companies to unforeseen risk of insolvencies. Since insurance, annuity and pension funds heavily depend on life tables when conducting actuarial valuations for policies and premium computations, there is a dire need for accurate and reliable mortality tables.

Grave concern in actuarial practice is the estimation problem relating to nonlinear mortality intensities that have been functionally formulated to obtain life insurance products. In related studies in [5, 6] it was observed that the uniform distribution of deaths interpolation $l_{x+t} = (1-t)l_x + tl_{x+1}$ has been specifically developed to solve most mortality estimation problems and expedite computational convenience while inadvertently trading off mathematical accuracy and consistencies. As a result, the same number of lives $l_x$ at age $x$ is expected to die periodically and because of this assumption, the authors applied the linear interpolation to estimate non-linear mortality functions in all actuarial computations but were unaware that the survival function $l_x \in C^K$.

From the foregoing, the following observations are evident: **(i)** Mortality functions based on linear interpolation are not analytically consistent with each other such that same mortality functions would mean different approximations between integral ages. (ii) Because of the identified functional inconsistencies, it can no longer be reasonably assumed that linear interpolation assumption is competent to estimate non-linear mortality functions. (iii) The assumption may not always be acceptable to a larger extent because equal number of lives cannot be assumed to die periodically. (iv) The second and higher derivatives of the survival function $l_x$ would be zero and hence renders the applications of both Taylor's and Euler-Maclaurin's series to mortality estimations inapplicable.

Arguably, further evidence of inconsistent mortality computations emerged in [7, 8] that the uniform distribution of death through linear interpolation assumption over which actuarial computations are currently based may no longer be adequate for many life insurance applications. Unknown in mathematics literature, some workers [9, 10] foresaw this problem in advance and constructed an interpolation resulting in mortality matrices which can be applied to model mortality rate intensity.

The deep investigation in [8] leading to the discovery of key computational deficiencies of linear interpolation was carried out to *expose* the inconsistencies in linear interpolation mortality assumption. The inadequate rate of mortality computations used in life insurance pricing and in mortality assumptions poses a serious challenge to life insurance operations. Unfortunately actuaries do not seem to observe this potential threat ahead and life insurance regulators similarly are being myopic of the possible dire consequences on the industry. The rising waves of research gaps therefore constitute critical concerns for life office to the extent that if they are not reasonably contained immediately can lead to insolvencies of life offices.

Let $l_x$ be a strictly decreasing function of real age such that if $x < y$ then $l_x > l_y$, the reason for this is that the insured lives must always die at an instant.

(ii) The number of deaths between ages $x$ and $x+1$ is defined as $d_x = l_x - l_{x+1}$. Since $l_x$ is strictly decreasing, $l_\Omega \to 0$ as no life exists towards the end of mortality table. The radix $l_0$ is the number of insured surviving at beginning of the mortality table.

**Differential equation governing the force of mortality $\mu_x$ at an instant**

According to [7, 11 – 14], the ratio of $-\dfrac{dl_{x+\zeta}}{d\zeta}$ to $l_{x+\zeta}$ at age $x$ and time $\zeta$ represents the mortality rate intensity. The ratio is the instantaneous force of mortality

$$\mu_x = -\lim_{\Delta\to 0}\frac{l_x - l_{x+\Delta}}{\Delta} \tag{5}$$

$$\mu_{x+\zeta} = -\frac{1}{l_{x+\zeta}}\frac{dl_{x+\zeta}}{d\zeta} = -\frac{d}{d\zeta}\ln l_{x+\zeta} \tag{6}$$

$$\frac{dl_{x+\zeta}}{d\zeta} = -\mu_{x+\zeta}l_{x+\zeta} \Rightarrow dl_{x+\zeta} = -\mu_{x+\zeta}l_{x+\zeta}d\zeta \tag{7}$$

Integrating (7) from age $0$ to age $\Omega - x$, we obtain

$$\int_0^{\Omega-x} dl_{x+\zeta} = -\int_0^{\Omega-x}\mu_{x+\zeta}l_{x+\zeta}d\zeta \tag{8}$$

$$\left[l_{x+\zeta}\right]_0^{\Omega-x} = -\int_0^{\Omega-x}\mu_{x+\zeta}l_{x+\zeta}d\zeta \tag{9}$$

Recall that $l_\Omega \to 0$ as no live survives at highest age $\Omega$ in the mortality table.

$$l_{x+\Omega-x} - l_x = -\int_0^{\Omega-x}\mu_{x+\zeta}l_{x+\zeta}d\zeta \tag{10}$$

The survivor's function at an arbitrary age $x$ is then obtained as

$$l_x = \int_0^{\Omega-x}\mu_{x+\zeta}l_{x+\zeta}d\zeta \tag{11}$$

Now, evaluating (9) from age $0$ to age $1$ and applying condition (ii) above

$$\left[l_{x+\zeta}\right]_0^1 = -\int_0^1\mu_{x+\zeta}l_{x+\zeta}d\zeta \Rightarrow l_{x+1} - l_x = -\int_0^1\mu_{x+\zeta}l_{x+\zeta}d\zeta \tag{12}$$

The number of deaths between ages $x$ and $x+1$ is obtained as

$$d_x = \int_0^1 \mu_{x+\zeta} l_{x+\zeta} d\zeta \qquad (13)$$

Again, evaluating (9) from age $0$ to age $\theta$ we have

$$\left[l_{x+\zeta}\right]_0^{\theta} = -\int_0^{\theta} \mu_{x+\zeta} l_{x+\zeta} d\zeta \Rightarrow l_{x+\theta} - l_x = -\int_0^{\theta} \mu_{x+\zeta} l_{x+\zeta} d\zeta \qquad (14)$$

The probability that a life aged $x$ dies before reaching age $x+\theta$ years is obtained by dividing (14) by $l_x$

$$ {}_{\theta}q_x = \int_0^{\theta} \mu_{x+\zeta} \frac{l_{x+\zeta}}{l_x} d\zeta = \int_0^{\theta} \mu_{x+\zeta} \left({}_{\zeta}p_x\right) d\zeta \qquad (15)$$

The severity to die is the hazard function $h_x(\theta) = \int_0^{\theta} \mu_{x+\zeta} d\zeta$ obtained by integrating both sides of (6) from time $0$ to age $\theta$,

$$\int_0^{\theta} \mu_{x+\zeta} d\zeta = -\int_0^{\theta} d\ln l_{x+\zeta} \qquad (16)$$

$$\int_0^{\theta} \mu_{x+\zeta} d\zeta = -\left(\ln l_{x+\theta} - \ln l_x\right) = -\ln \frac{l_{x+\theta}}{l_x} \qquad (17)$$

$$-\ln\left({}_{\theta}p_x\right) = \int_0^{\theta} \mu_{x+\zeta} d\zeta \qquad (18)$$

The probability that a life aged $x$ survives to the next age $(x+\theta)$ years becomes

$$\left({}_{\theta}p_x\right) = \exp\left(-\int_0^{\theta} \mu_{x+\zeta} d\zeta\right) \qquad (19)$$

as $\theta \to \infty$ in the result above, $\left({}_{\theta}p_x\right) = 0$ and the integral $\int_0^{\infty} \mu_{x+\zeta} d\zeta = \ln 0 = \infty$ while as $\theta \to 0$, $\left({}_{\theta}p_x\right) = 1$ and the integral $\int_0^{\theta} \mu_{x+\zeta} d\zeta = \ln 1 = 0$. Therefore the function $h_x(t)$ satisfies the conditions that $\lim_{\theta \to 0} h_x(\theta) = 0$ and $\lim_{\theta \to \infty} h_x(\theta) = \infty$.

The integral in equation (19) has the property that

$$\left({}_{\theta}p_x\right) = \exp\left(-\int_0^{\theta+x} \mu_{\zeta} d\zeta + \int_0^{x} \mu_{\zeta} d\zeta\right) \qquad (19a)$$

Letting $\zeta - x = u$, the survival probability becomes;

$$\left({}_{\theta}p_x\right) = \exp\left(-\int_{-x}^{\theta} \mu_{x+u} du + \int_{-x}^{0} \mu_{x+u} du\right) = \exp\left(-\int_{-x}^{0} \mu_{x+u} du - \int_{0}^{\theta} \mu_{x+u} du + \int_{-x}^{0} \mu_{x+u} du\right) \qquad (19b)$$

$$= e^{-\int_0^{\theta} \mu(x+u) du}$$

Differentiating the survival function (19) partially with respect to $t$, we obtain death density

$$\frac{\partial}{\partial t}\left({}_{t}p_x\right) = \frac{\partial}{\partial t} e^{-\int_0^{t} \mu_{x+\zeta} dt} = -\mu_{x+t} e^{-\int_0^{t} \mu_{x+\zeta} d\zeta} = -\mu_{x+t}\left({}_{t}p_x\right) = f_{T_x}(t) \qquad (20)$$

Therefore, from the arguments above, we state unequivocally the following results

$$\frac{d \ln l_x}{dx} = \lim_{\xi \to 0} \left( \frac{-\int_0^{\xi} e^{-\delta s}\left({}_{s}p_x\right) \mu_{x+s} ds}{\xi} \right) \qquad (21)$$

Applying L'Hopital's rule, equation (21) becomes

$$\frac{d \ln l_x}{dx} = -\lim_{\xi \to 0} \left( \frac{d}{d\xi} \left\{ \int_0^{\xi} e^{-\delta s}\left({}_{s}p_x\right) \mu_{x+s} ds \right\} \right) \qquad (22)$$

$$\frac{d \ln l_x}{dx} = -\lim_{\xi \to 0} \left[ e^{-\delta \xi}\left({}_{\xi}p_x\right) \mu_{x+\xi} \right] = -\mu_x \qquad (23)$$

The distribution and the complementary functions of $T_x$ are $F_{T_x}(s) = \int_0^{s} f_{T_x}(\zeta) d\zeta$ and

$$S_{T_x}(s) = 1 - F_{T_x}(s) \qquad (24)$$

However, in [15, 16], the continuous death density is in other words defined as

$$f_{T_x}(s) = \frac{d}{ds}\left({}_{s}q_x\right) \qquad (25)$$

The distribution of $T_x$ can be actuarially expressed as

$$F_{T_x}(s) = \mathbf{P}\left(T_x \le s\right) = {}_{s}q_x \qquad (26)$$

and

$$S_{T_x}(s) = \mathbf{P}\left(T_x > s\right) \qquad (27)$$

## Materials and Methods

**The method of severity to die function**

The severity to die in form integrated hazard function is required to model the probability of survival function. In order to account for increase in background mortality, we set $m=2$ and $n=2$ in (4) to obtain the mortality function (28a) where $A,B,C,H$ are mortality parameters satisfying law of parsimony (Occam's razor)

$$\mu_x = A + Hx + BC^x \qquad (28a)$$

Integrating both sides of (28a) from zero to an arbitrary age $x$ of the insured to obtain the severity to die function, we have

$$\int_0^x \mu_t dt = \int_0^x \left(A + Ht + BC^t\right) dt \qquad (28b)$$

$$\int_0^x \mu(t)dt = \left[At + \frac{Ht^2}{2} + \frac{BC^t}{\log_e C}\right]_0^x \qquad (28c)$$

$$\int_0^x \mu(t)dt = Ax + \frac{Hx^2}{2} + \frac{BC^x}{\log_e C} - \frac{B}{\log_e C} \qquad (28d)$$

In order to simplify (28d), we use the following transformations

$$\begin{cases} A = \log_e S^{-1} = (-1)\log_e S \\ H = \log_e W^{-2} = -2\log_e W \\ B = -(\log_e C)(\log_e g) \\ \log_e g = \dfrac{-B}{\log_e C} \\ K = \dfrac{l_0}{g} \end{cases} \qquad (28e)$$

Substituting the transformations in (28e) into (28d) we obtain;

$$\int_0^x \mu(t)dt = -(\log_e s)x - (\log_e W)x^2 - \frac{(\log_e C)(\log_e g)C^x}{\log_e C} + \frac{(\log_e C)(\log_e g)}{\log_e C} \qquad (28f)$$

Simplifying (28f), we have

$$\int_0^x \mu(t)dt = -(\log_e s)x - (\log_e W)x^2 - (\log_e g)C^x + (\log_e g) \qquad (28g)$$

$$\int_0^x \mu(t)dt = -x\log_e s - x^2\log_e W - C^x\log_e g + \log_e g \qquad (28h)$$

$$\int_0^x \mu(t)dt = -\log_e s^x - \log_e W^{x^2} - \log_e g^{C^x} + \log_e g \qquad (28i)$$

The survival probability function is given by

$${}_t p_x = \frac{l_{x+t}}{l_x} = \exp\left(-\int_0^t \mu_{x+\theta} d\theta\right) \qquad (28j)$$

In order to obtain the exponential function of the LHS in (28i), we set $x=0$ in (28j) as follows

$${}_t p_0 = \exp\left(-\int_0^t \mu_\theta d\theta\right) \qquad (28k)$$

$$l_t = l_0 \exp\left(-\int_0^t \mu_\theta d\theta\right) \qquad (28l)$$

The subscript $t$ in (28j) is an arbitrary age and consequently, we replace $t$ by insured's age $x$

$$l_x = l_0 \exp\left(-\int_0^x \mu_t dt\right) \qquad (28m)$$

Substituting (28i) into (28m) and have

$$l_x = l_0 e^{-\left[-\log_e s^x - \left(\log_e W^{x^2}\right) - \log_e g^{C^x} + \log_e g\right]} = l_0 e^{\left[\log_e s^x + \log_e W^{x^2} + \log_e g^{C^x} - \log_e g\right]} \qquad (28n)$$

Simplifying (28n) and obtain

$$l_x = l_0 e^{\left[\log_e\left(\frac{s^x W^{x^2} g^{C^x}}{g}\right)\right]} = l_0 \frac{s^x W^{x^2} g^{C^x}}{g} \qquad (28o)$$

The number of lives surviving to age $x$ is given as

$$l_x = \frac{l_0}{g} s^x W^{x^2} g^{C^x} = K s^x W^{x^2} g^{C^x}; \quad K = \frac{l_0}{g} \qquad (28p)$$

Taking logarithms of (28p) and have

$$\log_e l_x = \log_e K + \log_e s^x + \log_e W^{x^2} + \log_e g^{C^x} \qquad (29)$$

$$\log_e l_x = (\log_e K) + x(\log_e s) + x^2(\log_e W) + C^x(\log_e g) \qquad (30)$$

We need to obtain the values of $\phi=\left\{\overline{A};\overline{B};\overline{E};\overline{F}\right\}$ first, hence we use another transformation

Let

$Y=\log_e l_x;\overline{A}=\log_e K;\overline{B}=\log_e s;\overline{E}=\log_e W;\overline{F}=\log_e g$ (31)

Then

$$e^Y=l_x;e^{\overline{A}}=K;e^{\overline{B}}=S;e^{\overline{E}}=W;e^{\overline{F}}=g \quad (32)$$

$$l_x=e^{\overline{A}}e^{\overline{B}x}e^{\overline{E}x^2}e^{\overline{F}C^x}=e^{\overline{A}+\overline{B}x+\overline{E}x^2+\overline{F}C^x} \quad (33)$$

Putting (31) in (30) and obtain

$$Y=\overline{A}+\overline{B}x+\overline{E}x^2+\overline{F}C^x \quad (34)$$

substituting arbitrary five ages $x_1,x_2,x_3,x_4,x_5$ in(34) and using equal step lengths $x_2-x_1=x_3-x_2=x_4-x_3=x_5-x_4=h$, we have,

$$x_2=x_1+h;x_3=x_1+2h;x_4=x_1+3h;x_5=x_1+4h \quad (35)$$

$$Y_1=\overline{A}+\overline{B}x_1+\overline{E}x_1^{\,2}+\overline{F}C^{x_1} \quad (36)$$

$$Y_2=\overline{A}+\overline{B}x_2+\overline{E}x_2^{\,2}+\overline{F}C^{x_2} \quad (37)$$

$$Y_3=\overline{A}+\overline{B}x_3+\overline{E}x_3^{\,2}+\overline{F}C^{x_3} \quad (38)$$

$$Y_4=\overline{A}+\overline{B}x_4+\overline{E}x_4^{\,2}+\overline{F}C^{x_4} \quad (39)$$

$$Y_5=\overline{A}+\overline{B}x_5+\overline{E}x_5^{\,2}+\overline{F}C^{x_5} \quad (40)$$

Subtracting (36) from (37)

$$\Delta Y_1=Y_2-Y_1=\overline{A}+\overline{B}x_2+\overline{E}x_2^{\,2}+\overline{F}C^{x_2}-\left(\overline{A}+\overline{B}x_1+\overline{E}x_1^{\,2}+\overline{F}C^{x_1}\right) \quad (41)$$

$$\Delta Y_1=\overline{A}+\overline{B}x_2+\overline{E}x_2^{\,2}+\overline{F}C^{x_2}-\overline{A}-\overline{B}x_1-\overline{E}x_1^{\,2}-\overline{F}C^{x_1} \quad (42)$$

$$\Delta Y_1=\overline{B}\left(x_2-x_1\right)+\overline{E}\left(x_2^{\,2}-x_1^{\,2}\right)+\overline{F}\left(C^{x_2}-C^{x_1}\right) \quad (43)$$

$$\Delta Y_1=\overline{B}\left(x_2-x_1\right)+\overline{E}\left(x_2-x_1\right)\left(x_2+x_1\right)+\overline{F}\left(C^{x_2}-C^{x_1}\right) \quad (44)$$

Substituting $x_2$ defined in (35) into (44) we have

$$Y_2-Y_1=h\overline{B}+h\overline{E}\left(2x_1+h\right)+\overline{F}C^{x_1}\left(C^h-1\right) \quad (45)$$

Taking second difference between (38) and (37)

$$\Delta Y_2=Y_3-Y_2=\overline{A}+\overline{B}x_3+\overline{E}x_3^{\,2}+\overline{F}C^{x_3}-\left(\overline{A}+\overline{B}x_2+\overline{E}x_2^{\,2}+\overline{F}C^{x_2}\right) \quad (46)$$

$$\Delta Y_2=\overline{B}\left(x_3-x_2\right)+\overline{E}\left(x_3^{\,2}-x_2^{\,2}\right)+\overline{F}\left(C^{x_3}-C^{x_2}\right) \quad (47)$$

$$\Delta Y_2=h\overline{B}+\overline{E}\left(x_3-x_2\right)\left(x_3+x_2\right)+\overline{F}\left(C^{x_3}-C^{x_2}\right) \quad (48)$$

Following same procedure as before and substitute for $x_3$ defined in (35)

$$\Delta Y_2=h\overline{B}+h\overline{E}\left(x_1+2h+x_1+h\right)+\overline{F}\left(C^{x_1+2h}-C^{x_1+h}\right) \quad (49)$$

$$Y_3-Y_2=h\overline{B}+h\overline{E}\left(2x_1+3h\right)+\overline{F}C^{x_1+h}\left(C^h-1\right) \quad (50)$$

Taking third difference using (38) and (39) and substitute for $x_4$

$$\Delta Y_3=Y_4-Y_3=\overline{A}+\overline{B}x_4+\overline{E}x_4^{\,2}+\overline{F}C^{x_4}-\left(\overline{A}+\overline{B}x_3+\overline{E}x_3^{\,2}+\overline{F}C^{x_3}\right) \quad (51)$$

$$\Delta Y_3=h\overline{B}+h\overline{E}\left(x_1+3h+x_1+2h\right)+\overline{F}\left(C^{x_1+3h}-C^{x_1+2h}\right) \quad (52)$$

$$Y_4-Y_3=h\overline{B}+h\overline{E}\left(2x_1+5h\right)+\overline{F}C^{x_1+2h}\left(C^h-1\right) \quad (53)$$

Taking fourth difference using (39) and (40) and substitute for $x_5$

$$\Delta Y_4=Y_5-Y_4=\overline{A}+\overline{B}x_5+\overline{E}x_5^{\,2}+\overline{F}C^{x_5}-\left(\overline{A}+\overline{B}x_4+\overline{E}x_4^{\,2}+\overline{F}C^{x_4}\right) \quad (54)$$

$$\Delta Y_4=\overline{B}\left(x_5-x_4\right)+\overline{E}\left(x_5^{\,2}-x_4^{\,2}\right)+\overline{F}\left(C^{x_5}-C^{x_4}\right) \quad (55)$$

$$\Delta Y_4=\overline{B}\left(x_5-x_4\right)+\overline{E}\left(x_5-x_4\right)\left(x_5+x_4\right)+\overline{F}\left(C^{x_5}-C^{x_4}\right) \quad (56)$$

$$\Delta Y_4=h\overline{B}+h\overline{E}\left(x_1+4h+x_1+3h\right)+\overline{F}\left(C^{x_1+4h}-C^{x_1+3h}\right) \quad (57)$$

$$Y_5-Y_4=h\overline{B}+h\overline{E}\left(2x_1+7h\right)+\overline{F}C^{x_1+3h}\left(C^h-1\right) \quad (58)$$

subtracting (45) from (50) and obtain

$$Y_3 - Y_2 - (Y_2 - Y_1) = h\overline{B} + h\overline{E}(2x_1 + 3h) + \overline{F}C^{x_1+h}(C^h - 1) - \begin{pmatrix} h\overline{B} + h\overline{E}(2x_1 + h) \\ +\overline{F}C^{x_1}(C^h - 1) \end{pmatrix} \quad (59)$$

$$Y_3 - Y_2 - Y_2 + Y_1 = h\overline{B} + h\overline{E}(2x_1 + 3h) + \overline{F}C^{x_1+h}(C^h - 1) - h\overline{B} - h\overline{E}(2x_1 + h) - \overline{F}C^{x_1}(C^h - 1) \quad (60)$$

$$Y_3 - 2Y_2 + Y_1 = h\overline{B} - h\overline{B} + h\overline{E}(2x_1 + 3h) - h\overline{E}(2x_1 + h) + \overline{F}C^{x_1+h}(C^h - 1) - \overline{F}C^{x_1}(C^h - 1) \quad (61)$$

$$Y_3 - 2Y_2 + Y_1 = 2h^2\overline{E} + \overline{F}C^{x_1}(C^h - 1)^2 \quad (62)$$

Subtracting (50) from (53) and have

$$Y_4 - Y_3 - (Y_3 - Y_2) = h\overline{B} + h\overline{E}(2x_1 + 5h) + \overline{F}C^{x_1+2h}(C^h - 1) - \begin{bmatrix} h\overline{B} + h\overline{E}(2x_1 + 3h) \\ +\overline{F}C^{x_1+h}(C^h - 1) \end{bmatrix} \quad (63)$$

$$Y_4 - Y_3 - (Y_3 - Y_2) = h\overline{B} + h\overline{E}(2x_1 + 5h) + \overline{F}C^{x_1+2h}(C^h - 1) - h\overline{B} - h\overline{E}(2x_1 + 3h) - \overline{F}C^{x_1+h}(C^h - 1) \quad (64)$$

$$Y_4 - 2Y_3 + Y_2 = h\overline{B} - h\overline{B} + h\overline{E}(2x_1 + 5h) - h\overline{E}(2x_1 + 3h) + \overline{F}C^{x_1+2h}(C^h - 1) - \overline{F}C^{x_1+h}(C^h - 1) \quad (65)$$

$$Y_4 - 2Y_3 + Y_2 = 2h\overline{E}x_1 + 5h^2\overline{E} - 2h\overline{E}x_1 - 3h^2\overline{E} + \overline{F}C^{x_1+h}(C^h - 1)^2 \quad (66)$$

$$Y_4 - 2Y_3 + Y_2 = 2h^2\overline{E} + \overline{F}C^{x_1+h}(C^h - 1)^2 \quad (67)$$

Subtract (53) from (58) and have

$$Y_5 - Y_4 - (Y_4 - Y_3) = h\overline{B} + h\overline{E}(2x_1 + 7h) + \overline{F}C^{x_1+3h}(C^h - 1) - \begin{bmatrix} h\overline{B} + h\overline{E}(2x_1 + 5h) \\ +\overline{F}C^{x_1+2h}(C^h - 1) \end{bmatrix} \quad (68)$$

$$Y_5 - Y_4 - (Y_4 - Y_3) = h\overline{B} + h\overline{E}(2x_1 + 7h) + \overline{F}C^{x_1+3h}(C^h - 1) - h\overline{B} - h\overline{E}(2x_1 + 5h) - \overline{F}C^{x_1+2h}(C^h - 1) \quad (69)$$

$$Y_5 - Y_4 - (Y_4 - Y_3) = h\overline{B} - h\overline{B} + h\overline{E}(2x_1 + 7h) - h\overline{E}(2x_1 + 5h) + \overline{F}C^{x_1+3h}(C^h - 1) - \overline{F}C^{x_1+2h}(C^h - 1) \quad (70)$$

$$Y_5 - 2Y_4 + Y_3 = 2h\overline{E}x_1 + 7h^2\overline{E} - 2h\overline{E}x_1 - 5h^2\overline{E} + \overline{F}C^{x_1+2h}(C^h - 1)^2 \quad (71)$$

$$Y_5 - 2Y_4 + Y_3 = 2h^2\overline{E} + \overline{F}C^{x_1+2h}(C^h - 1)^2 \quad (72)$$

Subtract (62) from (67) and have

$$Y_4 - 2Y_3 + Y_2 - (Y_3 - 2Y_2 + Y_1) = 2h^2\overline{E} + \overline{F}C^{x_1+h}(C^h - 1)^2 - \left(2h^2\overline{E} + \overline{F}C^{x_1}(C^h - 1)^2\right) \quad (73)$$

$$Y_4 - 2Y_3 + Y_2 - Y_3 + 2Y_2 - Y_1 = 2h^2\overline{E} + \overline{F}C^{x_1+h}(C^h - 1)^2 - 2h^2\overline{E} - \overline{F}C^{x_1}(C^h - 1)^2 \quad (74)$$

$$Y_4 - 3Y_3 + 3Y_2 - Y_1 = \overline{F}C^{x_1}(C^h - 1)^3 \quad (75)$$

Subtract (67) from (72) and obtain

$$Y_5 - 2Y_4 + Y_3 - (Y_4 - 2Y_3 + Y_2) = 2h^2\overline{E} + \overline{F}C^{x_1+2h}(C^h - 1)^2 - \left[2h^2\overline{E} + \overline{F}C^{x_1+h}(C^h - 1)^2\right] \quad (76)$$

$$Y_5 - 2Y_4 + Y_3 - (Y_4 - 2Y_3 + Y_2) = 2h^2\overline{E} + \overline{F}C^{x_1+2h}(C^h - 1)^2 - 2h^2\overline{E} - \overline{F}C^{x_1+h}(C^h - 1)^2 \quad (77)$$

$$Y_5 - 2Y_4 + Y_3 - Y_4 + 2Y_3 - Y_2 = 2h^2\overline{E} + \overline{F}C^{x_1+2h}(C^h - 1)^2 - 2h^2\overline{E} - \overline{F}C^{x_1+h}(C^h - 1)^2 \quad (78)$$

$$Y_5 - 3Y_4 + 3Y_3 - Y_2 = \overline{F}C^{x_1+h}(C^h - 1)^3 \quad (79)$$

combining equations (75) and (79) together again and have

$$\begin{cases} Y_4 - 3Y_3 + 3Y_2 - Y_1 = \overline{F}C^{x_1}\left(C^h - 1\right)^3 \\ Y_5 - 3Y_4 + 3Y_3 - Y_2 = \overline{F}C^{x_1+h}\left(C^h - 1\right)^3 \end{cases} \tag{80}$$

Dividing the lower equation by the upper in system (80) to obtain the ageing parameter $C$

$$C^h = \frac{Y_5 - 3Y_4 + 3Y_3 - Y_2}{Y_4 - 3Y_3 + 3Y_2 - Y_1} \tag{81}$$

Setting (81) in (79) and have

$$Y_5 - 3Y_4 + 3Y_3 - Y_2 = \overline{F}\left[\left(\frac{Y_5 - 3Y_4 + 3Y_3 - Y_2}{Y_4 - 3Y_3 + 3Y_2 - Y_1}\right)^{\frac{x_1+h}{h}}\right]\left[\frac{Y_5 - 3Y_4 + 3Y_3 - Y_2}{Y_4 - 3Y_3 + 3Y_2 - Y_1} - 1\right]^3 \tag{82}$$

$$\overline{F}\left[\left(\frac{Y_5 - 3Y_4 + 3Y_3 - Y_2}{Y_4 - 3Y_3 + 3Y_2 - Y_1}\right)^{\frac{x_1+h}{h}}\right]\left[\frac{Y_5 - 3Y_4 + 3Y_3 - Y_2 - Y_4 + 3Y_3 - 3Y_2 + Y_1}{Y_4 - 3Y_3 + 3Y_2 - Y_1}\right]^3 = Y_5 - 3Y_4 + 3Y_3 - Y_2 \tag{83}$$

$$\overline{F}\left[\left(\frac{Y_5 - 3Y_4 + 3Y_3 - Y_2}{Y_4 - 3Y_3 + 3Y_2 - Y_1}\right)^{\frac{x_1+h}{h}}\right]\left[\frac{Y_5 - 4Y_4 + 6Y_3 - 4Y_2 + Y_1}{Y_4 - 3Y_3 + 3Y_2 - Y_1}\right]^3 = Y_5 - 3Y_4 + 3Y_3 - Y_2 \tag{84}$$

$$\overline{F} = \frac{Y_5 - 3Y_4 + 3Y_3 - Y_2}{\left[\left(\frac{Y_5 - 3Y_4 + 3Y_3 - Y_2}{Y_4 - 3Y_3 + 3Y_2 - Y_1}\right)^{\frac{x_1+h}{h}}\right]\left[\frac{Y_5 - 4Y_4 + 6Y_3 - 4Y_2 + Y_1}{Y_4 - 3Y_3 + 3Y_2 - Y_1}\right]^3} \tag{85}$$

Using the first equation in (62), we obtain

$$\overline{E} = \frac{Y_3 - 2Y_2 + Y_1 - \overline{F}C^{x_1}\left(C^h - 1\right)^2}{2h^2} \tag{86}$$

Recall from (45)

$$\overline{B} = \frac{Y_2 - Y_1 - h\overline{E}\left(2x_1 + h\right) - \overline{F}C^{x_1}\left(C^h - 1\right)}{h} \tag{87}$$

Using equation (36), we have

$$\overline{A} = Y_1 - \overline{B}x_1 - \overline{E}x_1^{\,2} - \overline{F}C^{x_1} \tag{88}$$

$$\bar{A} = Y_1 - \left\{ \left[ \frac{Y_2 - Y_1}{h} - (2x_1 + h)\left\{ \frac{Y_3 - 2Y_2 + Y_1}{2h^2} - \frac{1}{2h^2}\left\{ \frac{Y_5 - 3Y_4 + 3Y_3 - Y_2}{\left[\left(\frac{Y_5 - 3Y_4 + 3Y_3 - Y_2}{Y_4 - 3Y_3 + 3Y_2 - Y_1}\right)^{\frac{x_1+h}{h}}\right]\left[\frac{Y_5 - 4Y_4 + 6Y_3 - 4Y_2 + Y_1}{Y_4 - 3Y_3 + 3Y_2 - Y_1}\right]^3} \right\} \times \left\{ \left(\frac{Y_5 - 3Y_4 + 3Y_3 - Y_2}{Y_4 - 3Y_3 + 3Y_2 - Y_1}\right)^{\frac{x_1}{h}} \left[\frac{Y_5 - 4Y_4 + 6Y_3 - 4Y_2 + Y_1}{Y_4 - 3Y_3 + 3Y_2 - Y_1}\right]^2 \right\} \right\} - \frac{1}{h}\bar{F}\left[\left(\frac{Y_5 - 3Y_4 + 3Y_3 - Y_2}{Y_4 - 3Y_3 + 3Y_2 - Y_1}\right)^{\frac{x_1}{h}}\right]\left[\frac{Y_5 - 3Y_4 + 3Y_3 - Y_2}{Y_4 - 3Y_3 + 3Y_2 - Y_1} - 1\right] \right] \right\} x_1 - \left[ x_1^2 \left\{ \frac{Y_3 - 2Y_2 + Y_1}{2h^2} - \frac{1}{2h^2}\left\{ \frac{Y_5 - 3Y_4 + 3Y_3 - Y_2}{\left[\left(\frac{Y_5 - 3Y_4 + 3Y_3 - Y_2}{Y_4 - 3Y_3 + 3Y_2 - Y_1}\right)^{\frac{x_1+h}{h}}\right]\left[\frac{Y_5 - 4Y_4 + 6Y_3 - 4Y_2 + Y_1}{Y_4 - 3Y_3 + 3Y_2 - Y_1}\right]^3} \right\} \times \left\{ \left(\frac{Y_5 - 3Y_4 + 3Y_3 - Y_2}{Y_4 - 3Y_3 + 3Y_2 - Y_1}\right)^{\frac{x_1}{h}} \left[\frac{Y_5 - 3Y_4 + 3Y_3 - Y_2}{Y_4 - 3Y_3 + 3Y_2 - Y_1} - 1\right]^2 \right\} \right\} \right] - \bar{F}C^{x_1} \qquad (89)$$

where $\bar{F}$ and $C$ are given by (85) and (81)

The following auxiliary parameters $K; S; W; C; g$ can now be found from $\{\bar{A}; \bar{H}; \bar{B}, C, \bar{F}\}$ using the transformation $e^{Y} = l_x; e^{\bar{A}} = K; e^{\bar{B}} = S; e^{\bar{E}} = W; e^{\bar{F}} = g$. However, we can now compute the original mortality parameters $A; H; B$ using the transformations in (28c). Substituting the values into the force of mortality, equation (28a) is completely solved.

**Mortality experiments on the generalized Makeham's law for mortality intensities**

Proposition 1 (Claim): A life continues to be in a state of vitality or continuous longevity unless otherwise acted against his survival byan external force of mortality due to a source. Within the factors determining the evolution of mortality intensities, the human ageing characterizes the most pervasive effect in influencing the vulnerability ofmany causes of death. Consequently, age changes constitutes the highest demographic risk factor in allage related sicknesses.

It is on this argument that this study is based on age dependent mortality intensities satisfying Occam's razor and moreover, life insurance schemes issued for protection and pension purposes are defined on long or short term durations.

The data set employed in this study is a single year of age population data set $l_x$ taken from the published DAV 2008 German survival data for male.German data is considered because the country has proven mortality data collection records. Moreover Germany has death rate pattern similar to that of Nigeria. Germany has death rate of 12.3% while Nigeria has death rate of 13.1%. The ages considered inthe analysis are in single year within the age range0 –120. Our goal in this study is to compute the values of instantaneous mortality, curve of death, probability of death and probability of survival for a life insurance contract offered to a life aged $x$.

**Generalised Makeham's mortality life table**

In the Table that follows, $x$ represents the age, $l_x$ is the decreasing survival function, $\mu_x$ is the mortality rate intensity, $l_x\mu_x$ is the curve of death. $p_x$ is the probability that a life aged $x$ survives to the next age $x+1$, $q_x$ is the probability that a life aged $x$ will die before reaching age $x+1$, The mortality table below was generated using equations (28a), (81), (86), (87), (88).

## Results and Discussion

The study focuses on extended mathematical modelling and implementations of life table functions using Generalised Makeham's law of mortality. The findings are discussed along the male gender disparity.

In Table 1, the neonatal and infant mortality from birth to the first birthday is high but the mortality rate intensity falls rapidly within $0 \le x \le 7$ and becomes relatively stable within $8 \le x \le 9$ before attaining a minimum risk around age 10. The relative stability in this interval can be due to the predictable trend's gradual reduction in mortality over time or it can be due to idiosyncratic conditions neither due to shocks nor trends. This is an empirical evidence showing that the mortality rate generated through $GM(2,2)$ actually declines at 10. The parameters of $GM(2,2)$ is,

$$(A,\ B,\ H,\ C)_{MALE} = (0.003012821,\ 4.07194 \text{ x } 10^{-05},\ -0.000100466,\ 1.102923606) \qquad (90)$$

From the method of successive differencing employed to model $GM(2,2)$, the male ageing parameter values $1.102923606$ fall within the globally accepted interval $1.08 \le C \le 1.12$ for the $GM(m,n)$ family. This method is superior to the method of maximum likelihood estimation adopted in [17] where the ageing $C$ parameter is estimated as $1.024738$. The authors' method violated the permissible interval.

The mortality rate intensity further declines within $11 \le x \le 33$ and finally increases in the interval $34 \le x \le 120$. The possible medical intervention in the public health system to manage young adulthood mortality could be observed as the male's mortality rates have reduced. The decline in mortality rates within the former interval has significant impact on life annuities and on life insurers. However, the increase in mortality rates with age in the latter interval could be associated with various factors.

In men specifically, the risk of prostate cancer increases with age around $65$ and beyond until it is peaked between 78 and 79. Irrespective of sex, the probability of contracting a chronic sickness or disability increases with age while immunity reduces thus exposing lives more vulnerable to health risk and consequently increasing the risk of mortality.

As a result of deaths occurring over time, the survival curves for male in Figs 1 and 2 exhibit a consistent decrease in the number of lives surviving out of the initial 1000,000 newborns and consequently, their trajectories across ages display negative slope in line with equation, $-\frac{d}{dx}l_x = \mu_x$. This accounts for the reason why the curve $l_x$ is relatively flat for male with almost zero gradient. The curve exhibits a sudden sharp decline in the number of survivors around $80$ for male signifying that mortality rates are relatively low up till age $80$ before subsequently experiencing a sharp increase as observed in Fig. 1. The implication is that in Figs 1 and 2 the points at which the survival curve $l_x$ inflexes is around $80$ for both sexes. Furthermore, the curve of death in Fig. 1 tapers to a Mesokurtic curve. The mode of death intensity curve moves towards advanced ages and the concentration of death around the mode is increasing. Consequently, the survival curve $l_x$ is noted to exhibit a rectangularization pattern such that there is an observed progressively concentrated and increased mortality rates during senescent ages. As a result, the survival function assumes the observed rectangular shape. The accidental deaths at young ages are also seen to be increasing. These observed phenomena have pervasive socio-economic implications on the increasing number of lives attaining the retirement age as well as the period of extension during which the life annuity providers will pay out the benefits. Consequently, robust mortality rate estimation has a direct effect on the actuarial present value of future liabilities and the associated level of reserves that life office paying benefits holds. From age $106$ in Table 1, accurate estimation of mortality could be challenging due to errors in age heaping reporting from vital statistics and by the low number of lives and of deaths.

Particularly, the interval $107 \le x \le 120$ has the following implications. (i) Mortality data are scanty and are not reliable and as a result, the asymptotic behaviour of the underlying parsimonious function of mortality towards the end of mortality table is inconsistent. (ii) It seems many deaths were recorded but do not necessarily represent the point where $l_x$ inflexes.

**Table 1:** $GM(2,2)$ **male mortality table**

| $x$ | $l_x$ | $\mu_x$ | $l_x\mu_x$ | $q_x$ | $p_x$ |
|---|---|---|---|---|---|
| 0 | 1000000 | 0.00305354 | 3054 | 0.00300100 | 0.99699900 |
| 1 | 996999 | 0.00295727 | 2948 | 0.00290472 | 0.99709528 |
| 2 | 994103 | 0.00286142 | 2845 | 0.00280957 | 0.99719043 |
| 3 | 991310 | 0.00276605 | 2742 | 0.00271560 | 0.99728440 |
| 4 | 988618 | 0.00267121 | 2641 | 0.00261982 | 0.99738018 |
| 5 | 986028 | 0.00257695 | 2541 | 0.00252731 | 0.99747269 |
| 6 | 983536 | 0.00248332 | 2442 | 0.00243407 | 0.99756593 |
| 7 | 981142 | 0.00239040 | 2345 | 0.00234115 | 0.99765885 |
| 8 | 978845 | 0.00229825 | 2250 | 0.00225061 | 0.99774939 |
| 9 | 976642 | 0.00220696 | 2155 | 0.00215842 | 0.99784158 |
| 10 | 974534 | 0.00211662 | 2063 | 0.00207073 | 0.99792927 |
| 11 | 972516 | 0.00202731 | 1972 | 0.00198043 | 0.99801957 |
| 12 | 970590 | 0.00193916 | 1882 | 0.00189369 | 0.99810631 |
| 13 | 968752 | 0.00185227 | 1794 | 0.00180851 | 0.99819149 |
| 14 | 967000 | 0.00176678 | 1708 | 0.00172285 | 0.99827715 |
| 15 | 965334 | 0.00168283 | 1624 | 0.00163985 | 0.99836015 |
| 16 | 963751 | 0.00160059 | 1543 | 0.00155953 | 0.99844047 |
| 17 | 962248 | 0.00152021 | 1463 | 0.00147987 | 0.99852013 |
| 18 | 960824 | 0.00144191 | 1385 | 0.00140296 | 0.99859704 |
| 19 | 959476 | 0.00136588 | 1311 | 0.00132781 | 0.99867219 |
| 20 | 958202 | 0.00129238 | 1238 | 0.00125548 | 0.99874452 |
| 21 | 956999 | 0.00122164 | 1169 | 0.00118704 | 0.99881296 |
| 22 | 955863 | 0.00115397 | 1103 | 0.00112150 | 0.99887850 |
| 23 | 954791 | 0.00108967 | 1040 | 0.00105782 | 0.99894218 |
| 24 | 953781 | 0.00102909 | 982 | 0.00100023 | 0.99899977 |
| 25 | 952827 | 0.00097262 | 927 | 0.00094561 | 0.99905439 |
| 26 | 951926 | 0.00092068 | 876 | 0.00089608 | 0.99910392 |
| 27 | 951073 | 0.00087373 | 831 | 0.00085272 | 0.99914728 |
| 28 | 950262 | 0.00083229 | 791 | 0.00081346 | 0.99918654 |
| 29 | 949489 | 0.00079693 | 757 | 0.00078147 | 0.99921853 |
| 30 | 948747 | 0.00076826 | 729 | 0.00075679 | 0.99924321 |
| 31 | 948029 | 0.00074699 | 708 | 0.00073943 | 0.99926057 |
| 32 | 947328 | 0.00073386 | 695 | 0.00073153 | 0.99926847 |
| 33 | 946635 | 0.00072973 | 691 | 0.00073101 | 0.99926899 |
| 34 | 945943 | 0.00073551 | 696 | 0.00074317 | 0.99925683 |
| 35 | 945240 | 0.00075223 | 711 | 0.00076489 | 0.99923511 |
| 36 | 944517 | 0.00078100 | 738 | 0.00080041 | 0.99919959 |
| 37 | 943761 | 0.00082308 | 777 | 0.00084979 | 0.99915021 |
| 38 | 942959 | 0.00087983 | 830 | 0.00091520 | 0.99908480 |
| 39 | 942096 | 0.00095276 | 898 | 0.00099565 | 0.99900435 |
| 40 | 941158 | 0.00104354 | 982 | 0.00109652 | 0.99890348 |
| 41 | 940126 | 0.00115400 | 1085 | 0.00121792 | 0.99878208 |
| 42 | 938981 | 0.00128618 | 1208 | 0.00136105 | 0.99863895 |
| 43 | 937703 | 0.00144229 | 1352 | 0.00152927 | 0.99847073 |
| 44 | 936269 | 0.00162481 | 1521 | 0.00172707 | 0.99827293 |
| 45 | 934652 | 0.00183646 | 1716 | 0.00195367 | 0.99804633 |
| 46 | 932826 | 0.00208024 | 1941 | 0.00221478 | 0.99778522 |
| 47 | 930760 | 0.00235944 | 2196 | 0.00251193 | 0.99748807 |
| 48 | 928422 | 0.00267772 | 2486 | 0.00285000 | 0.99715000 |
| 49 | 925776 | 0.00303910 | 2814 | 0.00323404 | 0.99676596 |
| 50 | 922782 | 0.00344802 | 3182 | 0.00366826 | 0.99633174 |
| 51 | 919397 | 0.00390936 | 3594 | 0.00415490 | 0.99584510 |
| 52 | 915577 | 0.00442852 | 4055 | 0.00470305 | 0.99529695 |
| 53 | 911271 | 0.00501146 | 4567 | 0.00531785 | 0.99468215 |
| 54 | 906425 | 0.00566474 | 5135 | 0.00600491 | 0.99399509 |
| 55 | 900982 | 0.00639560 | 5762 | 0.00677372 | 0.99322628 |
| 56 | 894879 | 0.00721202 | 6454 | 0.00763008 | 0.99236992 |
| 57 | 888051 | 0.00812281 | 7213 | 0.00858397 | 0.99141603 |
| 58 | 880428 | 0.00913768 | 8045 | 0.00964531 | 0.99035469 |
| 59 | 871936 | 0.01026734 | 8952 | 0.01082534 | 0.98917466 |
| 60 | 862497 | 0.01152362 | 9939 | 0.01213569 | 0.98786431 |
| 61 | 852030 | 0.01291953 | 11008 | 0.01358755 | 0.98641245 |
| 62 | 840453 | 0.01446946 | 12161 | 0.01519776 | 0.98480224 |
| 63 | 827680 | 0.01618926 | 13400 | 0.01698120 | 0.98301880 |
| 64 | 813625 | 0.01809640 | 14724 | 0.01895345 | 0.98104655 |
| 65 | 798204 | 0.02021017 | 16132 | 0.02113370 | 0.97886630 |
| 66 | 781335 | 0.02255183 | 17621 | 0.02354560 | 0.97645440 |
| 67 | 762938 | 0.02514485 | 19184 | 0.02620527 | 0.97379473 |
| 68 | 742945 | 0.02801510 | 20814 | 0.02914348 | 0.97085652 |
| 69 | 721293 | 0.03119110 | 22498 | 0.03238212 | 0.96761788 |
| 70 | 697936 | 0.03470433 | 24221 | 0.03595315 | 0.96404685 |
| 71 | 672843 | 0.03858949 | 25965 | 0.03988300 | 0.96011700 |
| 72 | 646008 | 0.04288486 | 27704 | 0.04421307 | 0.95578693 |
| 73 | 617446 | 0.04763267 | 29411 | 0.04897270 | 0.95102730 |
| 74 | 587208 | 0.05287949 | 31051 | 0.05420566 | 0.94579434 |
| 75 | 555378 | 0.05867666 | 32588 | 0.05995196 | 0.94004804 |
| 76 | 522082 | 0.06508084 | 33978 | 0.06626162 | 0.93373838 |
| 77 | 487488 | 0.07215451 | 35174 | 0.07318129 | 0.92681871 |
| 78 | 451813 | 0.07996656 | 36130 | 0.08075908 | 0.91924092 |
| 79 | 415325 | 0.08859299 | 36795 | 0.08906037 | 0.91093963 |
| 80 | 378336 | 0.09811763 | 37121 | 0.09813235 | 0.90186765 |
| 81 | 341209 | 0.10863292 | 37067 | 0.10804522 | 0.89195478 |
| 82 | 304343 | 0.12024083 | 36594 | 0.11886260 | 0.88113740 |
| 83 | 268168 | 0.13305379 | 35681 | 0.13064944 | 0.86935056 |
| 84 | 233132 | 0.14719586 | 34316 | 0.14347666 | 0.85652334 |
| 85 | 199683 | 0.16280382 | 32509 | 0.15740949 | 0.84259051 |
| 86 | 168251 | 0.18002855 | 30290 | 0.17252201 | 0.82747799 |
| 87 | 139224 | 0.19903645 | 27711 | 0.18888985 | 0.81111015 |
| 88 | 112926 | 0.22001105 | 24845 | 0.20657776 | 0.79342224 |
| 89 | 89598 | 0.24315477 | 21786 | 0.22563004 | 0.77436996 |
| 90 | 69382 | 0.26869087 | 18642 | 0.24614453 | 0.75385547 |
| 91 | 52304 | 0.29686557 | 15527 | 0.26812481 | 0.73187519 |
| 92 | 38280 | 0.32795046 | 12554 | 0.29164054 | 0.70835946 |
| 93 | 27116 | 0.36224506 | 9823 | 0.31675026 | 0.68324974 |
| 94 | 18527 | 0.40007972 | 7412 | 0.34333675 | 0.65666325 |
| 95 | 12166 | 0.44181880 | 5375 | 0.37152721 | 0.62847279 |
| 96 | 7646 | 0.48786416 | 3730 | 0.40125556 | 0.59874444 |
| 97 | 4578 | 0.53865901 | 2466 | 0.43228484 | 0.56771516 |
| 98 | 2599 | 0.59469220 | 1546 | 0.46479415 | 0.53520585 |
| 99 | 1391 | 0.65650286 | 913 | 0.49820273 | 0.50179727 |
| 100 | 698 | 0.72468563 | 506 | 0.53295129 | 0.46704871 |
| 101 | 326 | 0.79989637 | 261 | 0.56748466 | 0.43251534 |
| 102 | 141 | 0.88285840 | 124 | 0.60283688 | 0.39716312 |
| 103 | 56 | 0.97436953 | 55 | 0.64285714 | 0.35714286 |
| 104 | 20 | 1.07530965 | 22 | 0.70000000 | 0.30000000 |
| 105 | 6 | 1.18664924 | 7 | 0.66666667 | 0.33333333 |
| 106 | 2 | 1.30945863 | 3 | 1.00000000 | 0.00000000 |
| 107 | 0 | 1.44491836 | 0 | 0.00000000 | 1.00000000 |
| 108 | 0 | 1.59433042 | 0 | 0.00000000 | 1.00000000 |
| 109 | 0 | 1.75913086 | 0 | 0.00000000 | 1.00000000 |
| 110 | 0 | 1.94090349 | 0 | 0.00000000 | 1.00000000 |
| 111 | 0 | 2.14139515 | 0 | 0.00000000 | 1.00000000 |
| 112 | 0 | 2.36253248 | 0 | 0.00000000 | 1.00000000 |
| 113 | 0 | 2.60644040 | 0 | 0.00000000 | 1.00000000 |
| 114 | 0 | 2.87546255 | 0 | 0.00000000 | 1.00000000 |
| 115 | 0 | 3.17218377 | 0 | 0.00000000 | 1.00000000 |
| 116 | 0 | 3.49945494 | 0 | 0.00000000 | 1.00000000 |
| 117 | 0 | 3.86042038 | 0 | 0.00000000 | 1.00000000 |
| 118 | 0 | 4.25854803 | 0 | 0.00000000 | 1.00000000 |
| 119 | 0 | 4.69766275 | 0 | 0.00000000 | 1.00000000 |
| 120 | 0 | 5.18198308 | 0 | 0.00000000 | 1.00000000 |

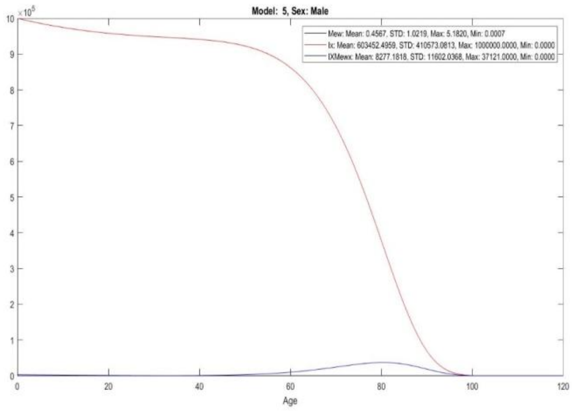


**Figure 1: *GM* (2,2) male survival and curve of death functions**

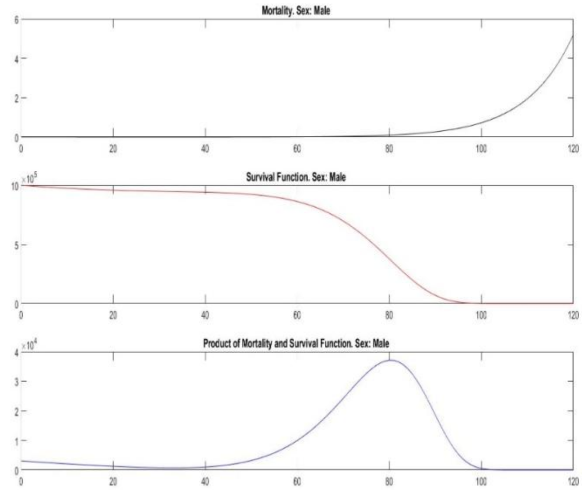


**Figure 2: *GM* (2,2) male's mortality, survival and curve of death functions**

The survival $l_x$ function in Figs. 1 and 2 representing the expected number of lives surviving to age $x$ out of an initial group of 1000000 lives clearly forms asymptote on $x$ axis.

In Table 1 and within $0 < x < 1$, the neonatal mortality rate intensity seems to be very high but declines more sharply for male.The probability density function of the distribution of deaths represents an important mortality statistics since it is an immediate indication of key longevity measures describing how long a population will live on the average and the extent of variability of ages at death. In Table 1 with respect to male, the curve of death function $l_x\mu_x$ describes the expected density of deaths at age $x$ with respect to the population of lives surviving to age $x$ and coincidentally for the male, there is a local minimum of $l_x\mu_x$ around $10$ where morality declines. The local extreme points of $l_x\mu_x$ corresponds to points of inflexion of $l_x$ following the observations below,

$$\frac{d}{dx}(l_x \times \mu_x) = -\frac{d}{dx}\left(l_x \times \frac{1}{l_x}\frac{d}{dx}l_x\right) = \frac{d}{dx}\left(-\frac{d}{dx}l_x\right) = -\frac{d^2}{dx^2}l_x \quad (92)$$

$$\frac{d}{dx}l_x\mu_x = \mu_x\frac{d}{dx}l_x + l_x\frac{d}{dx}\mu_x . \quad (93)$$

Observing that $l_x' = -\mu_x l_x$, we have

$$\frac{d}{dx}l_x\mu_x = \mu_x\left(-\mu_x l_x\right) + l_x\left(\mu_x'\right) \quad (94)$$

The point at which $l_x$ inflexes becomes

$$\frac{d}{dx}l_x\mu_x = \mu_x\left(-\mu_x l_x\right) + l_x\left(\mu_x'\right) = 0 \quad (95)$$

So that

$$\mu_x^2 = \frac{d}{dx}\mu_x \quad (96)$$

Since the functions $\{l_x, \mu_x, (l_x\mu_x)\}$ are functionally related, their behaviors are jointly examined through mortality surface and three dimensional plots as displayed in Fig. 3.

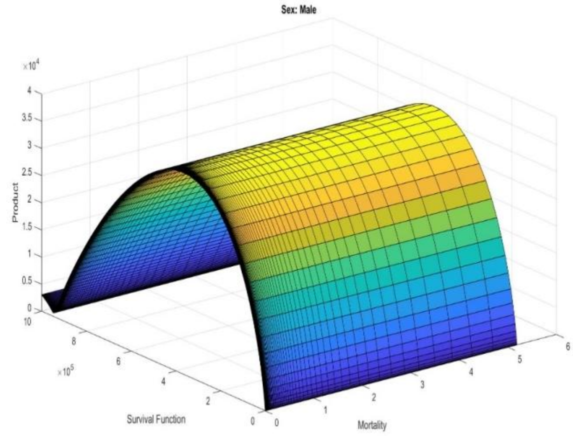


**Figure 3: *GM* (2,2) surface plot for male's mortality, survival and curve of death functions**

Notably, the function $\mu_x = GM(2,2) > 0$ is continuous for all $x \geq 0$ and satisfies the condition,

$$\int_0^\infty \mu_x dx = \infty . \quad (97)$$

Although in Table 1, the probability of death satisfies $0 < q_x \leq 1$, the mortality rate intensity is increasing and $\mu_x > 1$ towards the end of the mortality table. The implication in Table 1 is that there is a mortality risk of having a higher rate of death than expected. The high rate of mortality (jumps) may have occurred as a result of the sudden occurrence of pandemics or war. This is because $q_x$ is a probability whereas $\mu_x$ is a rate. Consequently at an infinitesimally small time $\Delta$ in

Table 1 during senescence, the mortality intensity is high and becomes,

$$\mu_x = -\frac{1}{l_x}\lim_{\Delta\to 0}\frac{l_{x+\Delta}-l_x}{\Delta} = \lim_{\Delta\to 0}\frac{\frac{l_x - l_{x+\Delta}}{l_x}}{\Delta} = \lim_{\Delta\to 0}\frac{{}_{\Delta}q_x}{\Delta} \quad (98)$$

where $0<\Delta<1$ and consequently, the above argument explains clearly the mathematical distinction between the mortality rate intensity $\mu_x$ and the probability of death $q_x$.

For instance if $\Delta = \frac{1}{3}$, then $\mu_x = 3\left({}_{\frac{1}{3}}q_x\right)$ and therefore, $\mu_x > 1$

The highest age in the mortality table is given by $\Omega = Sup\left\{\zeta \in \mathbf{R}^+ \middle| F_{T_x}(\zeta) \le 1\right\}$. Consequently, it is numerically determined from the estimated $l_x$ to be the first age where $q_x = 1$ whereas $\Omega_x(M) = 106$ for the males. To study the behaviour of mortality pattern beyond age $106$, the male's intensity is extended to age $120$ where the mortality rates at extreme age still exhibit exponential increase. This behaviour may not be valid in practice.

The omega age represents the age after which lives rarely survive because for all $x > \Omega$, $l_x = l_{x+1} = \ldots = 0$ and consequently, $\lim_{x\to 0} q_x = \lim_{x\to 0}\left(\frac{l_x - l_{x+1}}{l_x}\right) = \lim_{x\to 0}(1-p_x) = 0$. But the probability of death $q_x = 1$. These two arguments explain clear inconsistency of mortality behaviour in death probability function at extreme ages $q_x$ in Table 1. For the male in Table 1, the trajectories of $l_x$ is observed to decline steeply at perinatality till infancy and exhibiting at least a point of inflexion to $x$ axis as seen in Figure 1.

The argument that no life exists after age $106$ is supported as follows (i) The set $\Omega$ containing the domain of mortality validity is assumed to have no limit point (ii) then we can show that $\Omega$ is closed. $\Omega$ is closed $iff$ it contains all its limit points. Certainly by (i), $\Omega$ has no limit point by the hypothesis of the theorem, consequently, there are no limit point of $\Omega$ which are outside of $\Omega$ (which are not contained in $\Omega$), so $\Omega$ contains all its limit points. Therefore $\Omega$ is closed.

In Table 1, for male and within the age intervals $0 \le x \le 31$, $81 \le x \le 120$, we observe that $\mu_x > q_x$ however, within the interval $33 \le x \le 79$, $\mu_x < q_x$. By reason of equation (15), $l_x q_x = \int_0^1 l_{x+\xi}\mu_{x+\xi}d\xi$. If $l_{x+\xi}\mu_{x+\xi}$ were increasing, then $\frac{d}{d\xi}l_{x+\xi}\mu_{x+\xi} > 0$ and at the beginning of the interval $0<x<1$, $l_x q_x > l_x\mu_x \Rightarrow q_x > \mu_x$ provided the curve $l_x\mu_x$ is increasing. By definition, $-\frac{d}{dx}l_x = l_x\mu_x$, it then follows that when $l_x\mu_x$ is increasing, then clearly $\frac{d}{dx}l_x$ is decreasing and the gradient of the tangent to the curve $l_x$ will be decreasing. The survival function $l_x$ will then be concave to the age axis. Consequently, if the survival function $l_x$ is concave to the age axis, the condition $q_x > \mu_x$ is satisfied. However, if the survival function $l_x$ is convex to the age axis $q_x < \mu_x$. In Table 1 for male, $\mu_x \cong q_x$ almost at ages $\{32, 80\}$. The observation that $\mu_x \cong q_x$ in Table $1$ occurs whenever the survival function $l_x$ is almost linear in the form

$$\frac{l_x - l_{x+\xi}}{l_x - l_{x+1}} = \frac{x+\xi-x}{x+1-x} = \xi \quad (99)$$

for $0<\xi<1$ and

$$l_{x+\xi} = (1-\xi)l_x + \xi l_{x+1} \quad (100)$$

$$\mu_{x+\xi} = \frac{1}{l_{x+\xi}}\frac{dl_{x+\xi}}{d(x+\xi)} = \frac{1}{l_{x+\xi}}\frac{(l_{x+1}-l_x)}{d(x+\xi)} = \frac{(l_{x+1}-l_x)}{l_{x+\xi}} = \frac{(l_{x+1}-l_x)}{l_x}\times\frac{l_x}{l_{x+\xi}} \quad (101)$$

$$\mu_{x+\xi} = \frac{(l_{x+1}-l_x)}{l_x}\times\frac{1}{\frac{l_{x+\xi}}{l_x}} = \frac{(l_{x+1}-l_x)}{l_x}\times\frac{1}{{}_{\xi}p_x} = \frac{q_x}{1-\xi\times q_x} \quad (102)$$

Consequently at time $\xi = 0$, $q_x \cong \mu_x$.

The progressive increase in the mortality rate intensity for male is displayed in Fig. 1. Most life tables such as Commissioner's Standard Ordinary table (CSO) type and many published works do not account for $\mu_x$ based on the governing mortality intensities because of the computational intractability associated with their estimations. The male's ageing parameter our model is confirmed to satisfy the globally accepted ageing interval $1.08 < C < 1.12$. An interesting point in the modelling of $GM(2,2)$ is the novelty of the estimation technique through successive differencing applied. The technique gives better estimation for the ageing parameter than the maximum likelihood method (MLE) commonly used in most estimations.

The MLE technique does not often satisfy the globally accepted interval of validity and hence may be doomed in mortality analysis.When a life buys life insurance, the life is suspected to have a substandard health issues and hence such a life will have substandard mortality. As a result, life insurers will be interested in the functional relationship between standard and substandard mortality during the period of insurance. We assume here that the difference between standard and substandard mortality is the extra mortality risk which can be constant if $\Delta\mu_{x+s} - \Delta\mu_{x+s+1} = 0$, decreasing when $\Delta\mu_{x+s} - \Delta\mu_{x+s+1} > 0$ and increasing if $\Delta\mu_{x+s} - \Delta\mu_{x+s+1} < 0$

## Conclusion

In this study, we have investigated different age dependent mortality functions to generate with higher precision mortality rate intensities and life insurance products. The rationale behind the use of successive differencing approach is to ensure that mortality rate estimation is technically precise and appropriate to the mortality risk evaluation. Secondly in measuring the mortality intensities, the central problem is not concerned about the choice of the appropriate functional form which seems very important to our problem areas but rather a combination of analytical functions and actuarial assumptions that are critical in creating the pay-off space for the life insurance claim contingencies and life annuity benefits together with developing analytically robust life table models in order to solve the estimation problem of the non-linear mortality intensities so as to produce good underwriting results.The different numerical estimation methods and assumptions adopted in this study evolve from varying underwriting experiences emanating from: (i) various forms of life insurance valuation and underwriting techniques, (ii) the extent of their complexities and (iii) the level to which actuary usually advises life insurers on the methods to be applied when carrying out actuarial valuations.

Finally, our results suggest that the underlying functions applied such as successive differencing is well suited to the estimation of non-linear parsimonious mortality functions which satisfies the Occam's razor. If life offices do not apply accurate mortality table based on correct estimations, they will be vulnerable to the risk of paying much more death benefits than expected. Consequently life offices are therefore advised to reserve for mortality risks to mitigate against being insolvent.

**Conflict of interest/financial disclosure:** The authors declare no conflict of interest and no financial support was received for this study.

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**Citing this Article**

Ogungbenle, G. M., Sirisena, U. W., Chukwunenye, U. & Adeyele, J. S. (2024). Generating mortality rate intensity for life insurance applications through novel method of successive differencing under the parsimonious Generalised Makeham's framework. *Lafia Journal of Scientific and Industrial Research, 2*(2), 108 – 122. https://doi.org/10.62050/ljsir2024.v2n2.338