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# SOLVING OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS OF THE SECOND KIND USING BLOCK HYBRID METHOD

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## ABSTRACT
This study presented a Block Hybrid Method (BHM) for numerical integration of linear and nonlinear Volterra Integro-Differential Equations (VIDEs) of the second kind. The BHM was derived using the linear multistep method with the Linear Block Algorithm (LBA) to obtain a new block algorithm and its higher derivatives. Numerically, the analysis of the basic properties of the BHM was studied and found to be of uniform order seven, consistency, zero-stability, and convergence, with an A-stable region of absolute stability. The numerical simulation of BHM was carried out by integrating the BHM on linear and nonlinear Volterra integro-differential equations of second kind. The results obtained were compared with existing methods and presented in tables and graphically shown. The numerical comparisons show that the BHM produces highly accurate approximations and minimize the truncation error over the existing methods. The study shows that the BHM is efficient, stable and suitable for solving Volterra integro-differential equations of the second kind.

**Keywords:** BHM, VIDEs, Second kind, LBA, Consistency, Numerical simulation, Power series polynomial, Convergence analysis

## INTRODUCTION
Researchers, engineers, and technologists use integral equations as essential tools for modeling complex systems characterized by combined effects, interactions, and operational constraints. Integral equations are widely applied in physics, particularly in areas such as electrodynamics, fluid mechanics, and quantum mechanics, where they support the study of wave propagation, electromagnetic field distributions, and boundary integral-based analysis of material stresses (Aduroja *et al*., 2025; Otaide & Oluwayemi, 2024).
In engineering, integral equations serve as foundational techniques for a variety of applications, including signal processing, control systems, antenna design, and structural analysis through boundary element methods. In computer science, they play a significant role in fields such as image processing, artificial intelligence, and machine learning, where inverse problem techniques are used to reconstruct images from incomplete data (Oyedepo *et al*., 2024a). The integral equations are crucial to scientific and technological advancement, as they facilitate computational simulations and numerical modeling.
This study considers the numerical solution of Volterra integro-differential equations of second kind represented as

$$\rho^{(n)}(\xi)=\vartheta(\xi)+\varphi\int_0^{\xi}\mathrm{K}(\xi,\tau)\rho(\tau)d\tau \qquad (1.1)$$

Where $\varphi$ is a constant parameter, $\mathrm{K}(\xi,\tau)$ is called the kernel of integral equation, $\vartheta(\xi)$ is a function and $\xi$ and $0$ are the limits of integration can be constants. In equation (1.1), it is evident that the unknown function appears within the integral expression $\rho(\xi)$, the function to be determined typically appears both under the integral sign and in many cases, outside of it (Olowe *et al*., 2024).

The Volterra integro-differential equation (1.1) is widely used to model systems in which the current state depends on both present and past conditions, making it particularly valuable in applications such as population dynamics, epidemiological studies, and the analysis of viscoelastic materials (Oyedepo *et al*., 2024b). These equations combine differential and integral components, typically involving a kernel that represents memory effects within the system. However, obtaining analytical solutions to such equations is often challenging (Abdulganiy *et al*., 2018; Omole *et al*. 2024).

The Adomian Decomposition Method (ADM) and the Variational Iteration Method (VIM) are two widely used semi-analytical techniques for solving nonlinear differential equations. The ADM employs Adomian polynomials to handle nonlinear terms without the need for linearization, thereby enabling accurate and efficient solutions to complex problems, including higher-order equations and real-world applications such as COVID-19 modeling (Maturi & Malaikah, 2021; Agbata *et al*., 2022; Mak *et al*., 2021; Agarwala *et al*., 2022).

Similarly, the VIM generates rapidly convergent solutions through an iterative procedure that incorporates Lagrange multipliers for correction. The method can be further enhanced by hybrid or collocation approaches, which improve its efficiency and applicability to complex systems. However, its effectiveness depends on the selection of appropriate initial approximations and accurate determination of the multipliers (Patra, 2015; Memon *et al*., 2023; Liu *et al*., 2025; Rama *et al*., 2021). Both methods remain important tools for scientists and engineers in addressing theoretical and practical problems.

The Direct Computation Method (DCM) addresses Volterra integral equations by transforming them into systems of algebraic equations, which are then solved using appropriately selected quadrature methods. The incorporation of hybrid block-collocation techniques and analytical transforms further improves the method's stability and convergence, especially when dealing with complex problems (Oyedepo *et al*., 2024a; Rufai *et al*. 2023; Aggarwal *et al*., 2023; Wadi & Sulaiman, 2025).

The Linear Block Approach (,), on the other hand, is designed for solving initial value problems associated with ordinary differential equations. It operates by computing multiple solution points simultaneously through k-step block algorithms that utilize hybrid points to enhance accuracy and efficiency. This approach supports parallel computation and demonstrates strong performance in handling higher-order and oscillatory differential equations, making it widely applicable in scientific and engineering contexts (Adeyeye & Omar, 2016, 2019; Tumba *et al*., 2021; Raymond *et al*., 2023; Sabo & Adeyeye, 2025; Sunday *et al*. 2026).

**Derivation of Block Hybrid Method (BHM)**

The newly developed Block Hybrid Method (BHM) is derived from the Linear Block Algorithm (LBA) for the purpose of solving Volterra integro-differential equations of the second kind. The formulation of the BHM follows the procedures established in the LBA framework, as presented by Adeyeye and Omar in their 2016 and 2019 studies, where the method's foundation is supported by Proposition 2.1.

**Proposition 2.1:**

The linear multistep method of the form

$$\sum_{j=0}^{1}\alpha_j\rho_{n+j} = h^{\mu}\sum_{j=0}^{1}\beta_j\vartheta_{n+j} \tag{2.1}$$

exist only one numerical scheme from every BHM. The linear block algorithm of the form

$$\rho_{n+\eta} = \sum_{j=0}^{2}\frac{(\eta h)^j}{j!}\rho_n^{(j)} + \sum_{j=0}^{1}\left(\Lambda_{j\eta}\vartheta_{n+j}\right), \quad \eta = -\frac{1}{4}, -\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1 \tag{2.2}$$

and its higher derivatives of (2.1) of the form

$$\rho_{n+\eta}^{\sigma} = \sum_{j=0}^{2-\tau}\frac{(\eta h)^j}{j!}\rho_n^{(j+\sigma)} + \sum_{j=0}^{7}\left(X_{j\sigma}\,\vartheta_{n+j}\right), \sigma = 1_{\left(-\frac{1}{4}, -\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right)}, \sigma = 2_{\left(-\frac{1}{4}, -\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right)} \tag{2.3}$$

is consider, with $\Lambda_{\eta j} = \Psi^{-1}Z$ and $X_{\eta j\sigma} = \Psi^{-1}E$ where

$$\Psi=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{\left(-\frac{1}{4}h\right)^1}{1!} & \frac{\left(-\frac{1}{2}h\right)^1}{1!} & \frac{(0)^1}{1!} & \frac{\left(\frac{1}{4}h\right)^1}{1!} & \frac{\left(\frac{1}{2}h\right)^1}{1!} & \frac{\left(\frac{3}{4}h\right)^1}{1!} & \frac{(h)^1}{1!} \\ \frac{\left(-\frac{1}{4}h\right)^2}{2!} & \frac{\left(-\frac{1}{2}h\right)^2}{2!} & \frac{(0)^2}{2!} & \frac{\left(\frac{1}{4}h\right)^2}{2!} & \frac{\left(\frac{1}{2}h\right)^2}{2!} & \frac{\left(\frac{3}{4}h\right)^2}{2!} & \frac{(h)^2}{2!} \\ \frac{\left(-\frac{1}{4}h\right)^3}{3!} & \frac{\left(-\frac{1}{2}h\right)^3}{3!} & \frac{(0)^3}{3!} & \frac{\left(\frac{1}{4}h\right)^3}{3!} & \frac{\left(\frac{1}{2}h\right)^3}{3!} & \frac{\left(\frac{3}{4}h\right)^3}{3!} & \frac{(h)^3}{3!} \\ \frac{\left(-\frac{1}{4}h\right)^4}{4!} & \frac{\left(-\frac{1}{2}h\right)^4}{4!} & \frac{(0)^4}{4!} & \frac{\left(\frac{1}{4}h\right)^4}{4!} & \frac{\left(\frac{1}{2}h\right)^4}{4!} & \frac{\left(\frac{3}{4}h\right)^4}{4!} & \frac{(h)^4}{4!} \\ \frac{\left(-\frac{1}{4}h\right)^5}{5!} & \frac{\left(-\frac{1}{2}h\right)^5}{5!} & \frac{(0)^5}{5!} & \frac{\left(\frac{1}{4}h\right)^5}{5!} & \frac{\left(\frac{1}{2}h\right)^5}{5!} & \frac{\left(\frac{3}{4}h\right)^5}{5!} & \frac{(h)^5}{5!} \\ \frac{\left(-\frac{1}{4}h\right)^6}{6!} & \frac{\left(-\frac{1}{2}h\right)^6}{6!} & \frac{(0)^6}{6!} & \frac{\left(\frac{1}{4}h\right)^6}{6!} & \frac{\left(\frac{1}{2}h\right)^6}{6!} & \frac{\left(\frac{3}{4}h\right)^6}{6!} & \frac{(h)^6}{6!} \end{pmatrix},\ Z=\begin{pmatrix} \frac{(\xi h)^3}{3!} \\ \frac{(\xi h)^4}{4!} \\ \frac{(\xi h)^5}{5!} \\ \frac{(\xi h)^6}{6!} \\ \frac{(\xi h)^7}{7!} \\ \frac{(\xi h)^8}{8!} \\ \frac{(\xi h)^9}{9!} \end{pmatrix},\ E=\begin{pmatrix} \frac{(\xi h)^{3-\sigma}}{(3-\sigma)!} \\ \frac{(\xi h)^{4-\sigma}}{(4-\sigma)!} \\ \frac{(\xi h)^{5-\sigma}}{(5-\sigma)!} \\ \frac{(\xi h)^{6-\sigma}}{(6-\sigma)!} \\ \frac{(\xi h)^{7-\sigma}}{(7-\sigma)!} \\ \frac{(\xi h)^{8-\sigma}}{(8-\sigma)!} \\ \frac{(\xi h)^{9-\sigma}}{(9-\sigma)!} \end{pmatrix}$$

**Proof**

In order to obtain a BHM, equations (2.2) and (2.3) are solved sequentially to derive a polynomial of the form

$$\rho(\xi_n+\eta h)=\alpha_{\frac{1}{4}}\rho_{n+\frac{1}{4}}+\alpha_{\frac{1}{2}}\rho_{n+\frac{1}{2}}+\alpha_{\frac{3}{4}}\rho_{n+\frac{3}{4}}+h^3\left(\beta_{-\frac{1}{4}}\vartheta_{n-\frac{1}{4}}+\beta_{-\frac{1}{2}}\vartheta_{n-\frac{1}{2}}+\beta_0\vartheta_n+\beta_{\frac{1}{4}}\vartheta_{n+\frac{1}{4}}+\beta_{\frac{1}{2}}\vartheta_{n+\frac{1}{2}}+\beta_{\frac{3}{4}}\vartheta_{n+\frac{3}{4}}+\beta_1\vartheta_{n+1}\right) \quad (2.4)$$

Where $\eta=\xi_n+\xi h$ in the polynomial (2.4) and

$$\left.\begin{aligned}
&\alpha_{\frac{1}{4}}=3-10\eta+8\eta^2,\ \alpha_{\frac{1}{2}}=-3+16\eta-16\eta^2,\ \alpha_{\frac{3}{4}}=1-6\eta+8\eta^2,\\
&\beta_{-\frac{1}{4}}=-\frac{83}{645120}+\frac{713}{4838400}\eta+\frac{139}{40320}\eta^2-\frac{15}{15}\eta^4+\frac{28}{225}\eta^5-\frac{45}{45}\eta^6-\frac{16}{105}\eta^7+\frac{64}{315}\eta^8-\frac{64}{945}\eta^9\\
&\beta_{-\frac{1}{2}}=\frac{13}{1935360}-\frac{149}{4147200}\eta-\frac{289}{967680}\eta^2+\frac{1}{180}\eta^4-\frac{13}{1350}\eta^5-\frac{1}{90}\eta^6+\frac{8}{189}\eta^7-\frac{4}{105}\eta^8+\frac{32}{2835}\eta^9\\
&\beta_0=\frac{109}{322560}+\frac{7741}{1935360}\eta-\frac{5777}{107520}\eta^2+\frac{1}{6}\eta^3-\frac{7}{72}\eta^4-\frac{14}{45}\eta^5+\frac{7}{18}\eta^6+\frac{8}{45}\eta^7-\frac{4}{9}\eta^8+\frac{32}{189}\eta^9\\
&\beta_{\frac{1}{4}}=-\frac{7933}{967680}+\frac{94373}{1451520}\eta-\frac{17309}{120960}\eta^2+\frac{2}{9}\eta^4+\frac{4}{27}\eta^5-\frac{8}{15}\eta^6-\frac{32}{945}\eta^7+\frac{32}{63}\eta^8-\frac{128}{567}\eta^9\\
&\beta_{\frac{1}{2}}=-\frac{4927}{645120}+\frac{2495}{55296}\eta-\frac{17537}{322560}\eta^2-\frac{1}{12}\eta^4+\frac{1}{90}\eta^5+\frac{29}{90}\eta^6-\frac{8}{105}\eta^7+\frac{20}{63}\eta^8-\frac{32}{189}\eta^9\\
&\beta_{\frac{3}{4}}=\frac{1}{645120}+\frac{1313}{4838400}\eta-\frac{1}{480}\eta^2+\frac{1}{45}\eta^4-\frac{2}{225}\eta^5-\frac{4}{45}\eta^6+\frac{16}{315}\eta^7+\frac{32}{315}\eta^8-\frac{62}{945}\eta^9\\
&\beta_1=-\frac{1}{60480}+\frac{1837}{29030400}\eta+\frac{25}{193536}\eta^2-\frac{1}{360}\eta^4+\frac{1}{675}\eta^5+\frac{1}{90}\eta^6-\frac{8}{945}\eta^7-\frac{4}{315}\eta^8+\frac{32}{2835}\eta^9
\end{aligned}\right\} \quad (2.5)$$

The generalized algorithm (2.2) is expanded to give a BHM as

$$\left.\begin{aligned}
&\rho_{n-\frac{1}{4}}=\rho_n-\frac{1}{4}h\rho'_n+\frac{\left(-\frac{1}{4}h\right)^2}{2!}\rho''_n+h^3\left(\Lambda_{10}\vartheta_{n-\frac{1}{4}}+\Lambda_{11}\vartheta_{n-\frac{1}{2}}+\Lambda_{12}\vartheta_n+\Lambda_{13}\vartheta_{n+\frac{1}{4}}+\Lambda_{14}\vartheta_{n+\frac{1}{2}}+\Lambda_{15}\vartheta_{n+\frac{3}{4}}+\Lambda_{16}\vartheta_{n+1}\right)\\
&\rho_{n-\frac{1}{2}}=\rho_n-\frac{1}{2}h\rho'_n+\frac{\left(-\frac{1}{2}h\right)^2}{2!}\rho''_n+h^3\left(\Lambda_{20}\vartheta_{n-\frac{1}{4}}+\Lambda_{21}\vartheta_{n-\frac{1}{2}}+\Lambda_{22}\vartheta_n+\Lambda_{23}\vartheta_{n+\frac{1}{4}}+\Lambda_{24}\vartheta_{n+\frac{1}{2}}+\Lambda_{25}\vartheta_{n+\frac{3}{4}}+\Lambda_{26}\vartheta_{n+1}\right)\\
&\rho_{n+\frac{1}{4}}=\rho_n+\frac{1}{4}h\rho'_n+\frac{\left(\frac{1}{4}h\right)^2}{2!}\rho''_n+h^3\left(\Lambda_{30}\vartheta_{n-\frac{1}{4}}+\Lambda_{31}\vartheta_{n-\frac{1}{2}}+\Lambda_{32}\vartheta_n+\Lambda_{33}\vartheta_{n+\frac{1}{4}}+\Lambda_{34}\vartheta_{n+\frac{1}{2}}+\Lambda_{35}\vartheta_{n+\frac{3}{4}}+\Lambda_{36}\vartheta_{n+1}\right)\\
&\rho_{n+\frac{1}{2}}=\rho_n+\frac{1}{2}h\rho'_n+\frac{\left(\frac{1}{2}h\right)^2}{2!}\rho''_n+h^3\left(\Lambda_{40}\vartheta_{n-\frac{1}{4}}+\Lambda_{41}\vartheta_{n-\frac{1}{2}}+\Lambda_{42}\vartheta_n+\Lambda_{43}\vartheta_{n+\frac{1}{4}}+\Lambda_{44}\vartheta_{n+\frac{1}{2}}+\Lambda_{45}\vartheta_{n+\frac{3}{4}}+\Lambda_{46}\vartheta_{n+1}\right)\\
&\rho_{n+\frac{3}{4}}=\rho_n+\frac{3}{4}h\rho'_n+\frac{\left(\frac{3}{4}h\right)^2}{2!}\rho''_n+h^3\left(\Lambda_{50}\vartheta_{n-\frac{1}{4}}+\Lambda_{51}\vartheta_{n-\frac{1}{2}}+\Lambda_{52}\vartheta_n+\Lambda_{53}\vartheta_{n+\frac{1}{4}}+\Lambda_{54}\vartheta_{n+\frac{1}{2}}+\Lambda_{55}\vartheta_{n+\frac{3}{4}}+\Lambda_{56}\vartheta_{n+1}\right)\\
&\rho_{n+1}=\rho_n+h\rho'_n+\frac{(h)^2}{2!}\rho''_n+h^3\left(\Lambda_{60}\vartheta_{n-\frac{1}{4}}+\Lambda_{61}\vartheta_{n-\frac{1}{2}}+\Lambda_{62}\vartheta_n+\Lambda_{63}\vartheta_{n+\frac{1}{4}}+\Lambda_{64}\vartheta_{n+\frac{1}{2}}+\Lambda_{65}\vartheta_{n+\frac{3}{4}}+\Lambda_{66}\vartheta_{n+1}\right)
\end{aligned}\right\} \quad (2.6)$$

The higher derivatives of the BHM are

$$\left.\begin{aligned}
\rho'_{n-\frac{1}{4}} &= \rho'_n - \frac{1}{4}h\rho''_n + h^2 f\left(\mathrm{X}_{110}\vartheta_{n-\frac{1}{4}} + \mathrm{X}_{111}\vartheta_{n-\frac{1}{2}} + \mathrm{X}_{112}\vartheta_n + \mathrm{X}_{113}\vartheta_{n+\frac{1}{4}} + \mathrm{X}_{114}\vartheta_{n+\frac{1}{2}} + \mathrm{X}_{115}\vartheta_{n+\frac{3}{4}} + \mathrm{X}_{116}\vartheta_{n+1}\right)\\
\rho'_{n-\frac{1}{2}} &= \rho'_n - \frac{1}{2}h\rho''_n + h^2\left(\mathrm{X}_{120}\vartheta_{n-\frac{1}{4}} + \mathrm{X}_{121}\vartheta_{n-\frac{1}{2}} + \mathrm{X}_{122}\vartheta_n + \mathrm{X}_{123}\vartheta_{n+\frac{1}{4}} + \mathrm{X}_{124}\vartheta_{n+\frac{1}{2}} + \mathrm{X}_{125}\vartheta_{n+\frac{3}{4}} + \mathrm{X}_{126}\vartheta_{n+1}\right)\\
\rho'_{n+\frac{1}{4}} &= \rho'_n + \frac{1}{4}h\rho''_n + h^2\left(\mathrm{X}_{130}\vartheta_{n-\frac{1}{4}} + \mathrm{X}_{131}\vartheta_{n-\frac{1}{2}} + \mathrm{X}_{132}\vartheta_n + \mathrm{X}_{133}\vartheta_{n+\frac{1}{4}} + \mathrm{X}_{134}\vartheta_{n+\frac{1}{2}} + \mathrm{X}_{135}\vartheta_{n+\frac{3}{4}} + \mathrm{X}_{136}\vartheta_{n+1}\right)\\
\rho'_{n+\frac{1}{2}} &= \rho'_n + \frac{1}{2}h\rho''_n + h^2\left(\mathrm{X}_{140}\vartheta_{n-\frac{1}{4}} + \mathrm{X}_{141}\vartheta_{n-\frac{1}{2}} + \mathrm{X}_{142}\vartheta_n + \mathrm{X}_{143}\vartheta_{n+\frac{1}{4}} + \mathrm{X}_{144}\vartheta_{n+\frac{1}{2}} + \mathrm{X}_{145}\vartheta_{n+\frac{3}{4}} + \mathrm{X}_{146}\vartheta_{n+1}\right)\\
\rho'_{n+\frac{3}{4}} &= \rho'_n + \frac{3}{4}h\rho''_n + h^2\left(\mathrm{X}_{150}\vartheta_{n-\frac{1}{4}} + \mathrm{X}_{151}\vartheta_{n-\frac{1}{2}} + \mathrm{X}_{152}\vartheta_n + \mathrm{X}_{153}\vartheta_{n+\frac{1}{4}} + \mathrm{X}_{154}\vartheta_{n+\frac{1}{2}} + \mathrm{X}_{155}\vartheta_{n+\frac{3}{4}} + \mathrm{X}_{156}\vartheta_{n+1}\right)\\
\rho'_{n+1} &= \rho'_n + h\rho''_n + h^2\left(\mathrm{X}_{160}\vartheta_{n-\frac{1}{4}} + \mathrm{X}_{161}\vartheta_{n-\frac{1}{2}} + \mathrm{X}_{162}\vartheta_n + \mathrm{X}_{163}\vartheta_{n+\frac{1}{4}} + \mathrm{X}_{164}\vartheta_{n+\frac{1}{2}} + \mathrm{X}_{165}\vartheta_{n+\frac{3}{4}} + \mathrm{X}_{166}\vartheta_{n+1}\right)
\end{aligned}\right\} \qquad (2.7)$$

$$\left.\begin{aligned}
\rho''_{n-\frac{1}{4}} &= \rho''_n + h\left(\mathrm{X}_{210}\vartheta_{n-\frac{1}{4}} + \mathrm{X}_{211}\vartheta_{n-\frac{1}{2}} + \mathrm{X}_{212}\vartheta_n + \mathrm{X}_{213}\vartheta_{n+\frac{1}{4}} + \mathrm{X}_{214}\vartheta_{n+\frac{1}{2}} + \mathrm{X}_{215}\vartheta_{n+\frac{3}{4}} + \mathrm{X}_{216}\vartheta_{n+1}\right)\\
\rho''_{n-\frac{1}{2}} &= \rho''_n + h\left(\mathrm{X}_{220}\vartheta_{n-\frac{1}{4}} + \mathrm{X}_{221}\vartheta_{n-\frac{1}{2}} + \mathrm{X}_{222}\vartheta_n + \mathrm{X}_{223}\vartheta_{n+\frac{1}{4}} + \mathrm{X}_{224}\vartheta_{n+\frac{1}{2}} + \mathrm{X}_{225}\vartheta_{n+\frac{3}{4}} + \mathrm{X}_{226}\vartheta_{n+1}\right)\\
\rho''_{n+\frac{1}{4}} &= \rho''_n + h\left(\mathrm{X}_{230}\vartheta_{n-\frac{1}{4}} + \mathrm{X}_{231}\vartheta_{n-\frac{1}{2}} + \mathrm{X}_{232}\vartheta_n + \mathrm{X}_{233}\vartheta_{n+\frac{1}{4}} + \mathrm{X}_{234}\vartheta_{n+\frac{1}{2}} + \mathrm{X}_{235}\vartheta_{n+\frac{3}{4}} + \mathrm{X}_{236}\vartheta_{n+1}\right)\\
\rho''_{n+\frac{1}{2}} &= \rho''_n + h\left(\mathrm{X}_{240}\vartheta_{n-\frac{1}{4}} + \mathrm{X}_{241}\vartheta_{n-\frac{1}{2}} + \mathrm{X}_{242}\vartheta_n + \mathrm{X}_{243}\vartheta_{n+\frac{1}{4}} + \mathrm{X}_{244}\vartheta_{n+\frac{1}{2}} + \mathrm{X}_{245}\vartheta_{n+\frac{3}{4}} + \mathrm{X}_{246}\vartheta_{n+1} +\right)\\
\rho''_{n+\frac{3}{4}} &= \rho''_n + h\left(\mathrm{X}_{250}\vartheta_{n-\frac{1}{4}} + \mathrm{X}_{251}\vartheta_{n-\frac{1}{2}} + \mathrm{X}_{252}\vartheta_n + \mathrm{X}_{253}\vartheta_{n+\frac{1}{4}} + \mathrm{X}_{254}\vartheta_{n+\frac{1}{2}} + \mathrm{X}_{255}\vartheta_{n+\frac{3}{4}} + \mathrm{X}_{256}\vartheta_{n+1} +\right)\\
\rho''_{n+1} &= \rho''_n + h\left(\mathrm{X}_{260}\vartheta_{n-\frac{1}{4}} + \mathrm{X}_{261}\vartheta_{n-\frac{1}{2}} + \mathrm{X}_{262}\vartheta_n + \mathrm{X}_{263}\vartheta_{n+\frac{1}{4}} + \mathrm{X}_{264}\vartheta_{n+\frac{1}{2}} + \mathrm{X}_{265}\vartheta_{n+\frac{3}{4}} + \mathrm{X}_{266}\vartheta_{n+1}\right)
\end{aligned}\right\} \qquad (2.8)$$

By simplifying $\Lambda_{\eta j} = \Psi^{-1}\mathrm{Z}$, the unknown coefficients of $\Lambda$ in equation (2.7) are obtained as

$$\begin{pmatrix}\Lambda_{10}\\ \Lambda_{11}\\ \Lambda_{12}\\ \Lambda_{13}\\ \Lambda_{14}\\ \Lambda_{15}\\ \Lambda_{16}\end{pmatrix} = \begin{pmatrix}-\frac{1201}{2764800}\\ \frac{5849}{232243200}\\ -\frac{40309}{15482880}\\ \frac{1753}{2903040}\\ -\frac{4003}{15482880}\\ \frac{1403}{19353600}\\ -\frac{2161}{232243200}\end{pmatrix}, \begin{pmatrix}\Lambda_{20}\\ \Lambda_{21}\\ \Lambda_{22}\\ \Lambda_{23}\\ \Lambda_{24}\\ \Lambda_{25}\\ \Lambda_{26}\end{pmatrix} = \begin{pmatrix}-\frac{659}{75600}\\ -\frac{7}{259200}\\ -\frac{881}{60480}\\ \frac{41}{11340}\\ -\frac{181}{120960}\\ \frac{31}{75600}\\ -\frac{49}{907200}\end{pmatrix}, \begin{pmatrix}\Lambda_{30}\\ \Lambda_{31}\\ \Lambda_{32}\\ \Lambda_{33}\\ \Lambda_{34}\\ \Lambda_{35}\\ \Lambda_{36}\end{pmatrix} = \begin{pmatrix}-\frac{2393}{19353600}\\ \frac{2701}{232243200}\\ \frac{31279}{15482880}\\ \frac{23}{25920}\\ -\frac{3791}{15482880}\\ \frac{1177}{19353600}\\ -\frac{1709}{232243200}\end{pmatrix}, \begin{pmatrix}\Lambda_{40}\\ \Lambda_{41}\\ \Lambda_{42}\\ \Lambda_{43}\\ \Lambda_{44}\\ \Lambda_{45}\\ \Lambda_{46}\end{pmatrix} = \begin{pmatrix}-\frac{61}{75600}\\ \frac{139}{1814400}\\ \frac{671}{60480}\\ \frac{13}{1134}\\ -\frac{23}{17280}\\ \frac{29}{75600}\\ -\frac{43}{907200}\end{pmatrix}, \begin{pmatrix}\Lambda_{50}\\ \Lambda_{51}\\ \Lambda_{52}\\ \Lambda_{53}\\ \Lambda_{54}\\ \Lambda_{55}\\ \Lambda_{56}\end{pmatrix} = \begin{pmatrix}-\frac{1377}{716800}\\ \frac{513}{2867200}\\ \frac{15417}{573440}\\ \frac{1431}{35840}\\ \frac{2511}{573440}\\ \frac{99}{102400}\\ \frac{297}{2867200}\end{pmatrix}, \begin{pmatrix}\Lambda_{60}\\ \Lambda_{61}\\ \Lambda_{62}\\ \Lambda_{63}\\ \Lambda_{64}\\ \Lambda_{65}\\ \Lambda_{66}\end{pmatrix} = \begin{pmatrix}-\frac{17}{4725}\\ \frac{19}{56700}\\ \frac{377}{7560}\\ \frac{242}{2835}\\ \frac{97}{3780}\\ \frac{43}{4725}\\ -\frac{1}{16200}\end{pmatrix}$$

Similarly, the expressions $\mathrm{X}_{\eta j\sigma} = \Psi^{-1}\mathrm{E}$ are simplified to determine the unknown coefficients of the higher derivative $\mathrm{X}$ in equations (2.7) and (2.8) as:

$$\begin{pmatrix} X_{110} \\ X_{111} \\ X_{112} \\ X_{113} \\ X_{114} \\ X_{115} \\ X_{116} \end{pmatrix} = \begin{pmatrix} \frac{39}{5120} \\ -\frac{731}{1935360} \\ \frac{6347}{215040} \\ \frac{3971}{483840} \\ \frac{257}{71680} \\ \frac{109}{107520} \\ \frac{253}{1935360} \end{pmatrix}, \begin{pmatrix} X_{120} \\ X_{121} \\ X_{123} \\ X_{113} \\ X_{124} \\ X_{125} \\ X_{126} \end{pmatrix} = \begin{pmatrix} \frac{179}{2520} \\ \frac{1}{270} \\ \frac{257}{10080} \\ -\frac{31}{3780} \\ \frac{11}{5040} \\ -\frac{1}{2520} \\ \frac{1}{30240} \end{pmatrix}, \begin{pmatrix} X_{130} \\ X_{131} \\ X_{132} \\ X_{133} \\ X_{134} \\ X_{135} \\ X_{136} \end{pmatrix} = \begin{pmatrix} -\frac{503}{322560} \\ \frac{289}{1935360} \\ \frac{13861}{645120} \\ \frac{191}{13824} \\ -\frac{2171}{645120} \\ \frac{53}{64512} \\ -\frac{191}{1935360} \end{pmatrix}, \begin{pmatrix} X_{140} \\ X_{141} \\ X_{142} \\ X_{143} \\ X_{144} \\ X_{145} \\ X_{146} \end{pmatrix} = \begin{pmatrix} -\frac{1}{280} \\ \frac{1}{3024} \\ \frac{169}{3360} \\ \frac{293}{3780} \\ 0 \\ \frac{1}{840} \\ -\frac{1}{6048} \end{pmatrix}, \begin{pmatrix} X_{150} \\ X_{151} \\ X_{152} \\ X_{153} \\ X_{154} \\ X_{155} \\ X_{156} \end{pmatrix} = \begin{pmatrix} -\frac{207}{35840} \\ \frac{39}{71680} \\ \frac{5601}{71680} \\ \frac{2631}{17920} \\ \frac{3897}{71680} \\ \frac{39}{5120} \\ -\frac{33}{71680} \end{pmatrix}, \begin{pmatrix} X_{160} \\ X_{161} \\ X_{162} \\ X_{163} \\ X_{164} \\ X_{165} \\ X_{166} \end{pmatrix} = \begin{pmatrix} -\frac{2}{315} \\ \frac{1}{1890} \\ \frac{32}{315} \\ \frac{212}{945} \\ \frac{67}{630} \\ \frac{22}{315} \\ \frac{1}{270} \end{pmatrix}$$

$$\begin{pmatrix} X_{210} \\ X_{211} \\ X_{212} \\ X_{213} \\ X_{214} \\ X_{215} \\ X_{216} \end{pmatrix} = \begin{pmatrix} -\frac{349}{3360} \\ \frac{863}{241920} \\ -\frac{5221}{26880} \\ \frac{127}{1890} \\ -\frac{811}{26880} \\ \frac{29}{3360} \\ -\frac{271}{241920} \end{pmatrix}, \begin{pmatrix} X_{220} \\ X_{221} \\ X_{222} \\ X_{223} \\ X_{224} \\ X_{225} \\ X_{226} \end{pmatrix} = \begin{pmatrix} -\frac{47}{126} \\ -\frac{1139}{15120} \\ -\frac{11}{5040} \\ -\frac{83}{945} \\ \frac{269}{5040} \\ -\frac{11}{630} \\ \frac{37}{15120} \end{pmatrix}, \begin{pmatrix} X_{230} \\ X_{231} \\ X_{232} \\ X_{233} \\ X_{234} \\ X_{235} \\ X_{236} \end{pmatrix} = \begin{pmatrix} -\frac{23}{2016} \\ \frac{271}{241920} \\ \frac{10273}{80640} \\ \frac{293}{1890} \\ -\frac{2257}{80640} \\ \frac{67}{10080} \\ -\frac{191}{241920} \end{pmatrix}, \begin{pmatrix} X_{240} \\ X_{241} \\ X_{242} \\ X_{243} \\ X_{244} \\ X_{245} \\ X_{246} \end{pmatrix} = \begin{pmatrix} -\frac{1}{210} \\ \frac{1}{3024} \\ \frac{167}{1680} \\ \frac{293}{945} \\ \frac{167}{1680} \\ -\frac{1}{210} \\ \frac{1}{3024} \end{pmatrix}, \begin{pmatrix} X_{250} \\ X_{215} \\ X_{252} \\ X_{253} \\ X_{254} \\ X_{255} \\ X_{256} \end{pmatrix} = \begin{pmatrix} -\frac{3}{224} \\ \frac{13}{8960} \\ \frac{1161}{8960} \\ \frac{17}{70} \\ \frac{2631}{8960} \\ \frac{111}{1120} \\ -\frac{29}{8960} \end{pmatrix}, \begin{pmatrix} X_{260} \\ X_{261} \\ X_{262} \\ X_{263} \\ X_{264} \\ X_{265} \\ X_{266} \end{pmatrix} = \begin{pmatrix} \frac{4}{315} \\ -\frac{2}{945} \\ \frac{29}{630} \\ \frac{376}{945} \\ \frac{32}{315} \\ \frac{116}{315} \\ \frac{149}{1890} \end{pmatrix}$$

**Analysis of the Basic Properties of Block Hybrid Method (BHM)**

The essential feature of the BHM was studied, including its full error constant, consistency, zero-state stability, convergence, and area of absolute stability.

**Order and Error Constant of BHM**

The linear operator $L[\rho(\eta_n);h]$ with the following corollary 3.1 and 3.2 to determining the order and error constant of the new method.

**Corollary 3.1**

i. The linear operator $L[\rho(\eta_n);h]$ associate with the local truncation error of the BHM and its higher derivatives (2.7) and (2.8) is

$C_{07}\,h^{07}y^{07}(t_n) + 0(h^{10}), C_{07}\,h^{07}\,y^{07}(t_n) + 0(h^{09}), C_{07}\,h^{07}\,y^{07}(t_n) + 0(h^{08})$ (Shwame *et al.*, 2024).

**Proof**

According to (Skwame *et al.*, 2024), the linear difference operators associated with the BHM are given by

$$\left.\begin{array}{l} L[\rho(\eta_n);h] = \rho_{n-\frac{1}{4}} + \rho_n + \frac{1}{4}h\rho'_n - \frac{\left(-\frac{1}{4}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{10}\vartheta_{n-\frac{1}{4}} + \Lambda_{11}\vartheta_{n-\frac{1}{2}} + \Lambda_{12}\vartheta_n + \Lambda_{13}\vartheta_{n+\frac{1}{4}} + \Lambda_{14}\vartheta_{n+\frac{1}{2}} + \Lambda_{15}\vartheta_{n+\frac{3}{4}} + \Lambda_{16}\vartheta_{n+1}\right) \\ L[\rho(\eta_n);h] - \rho_{n-\frac{1}{2}} - \rho_n + \frac{1}{2}h\rho'_n - \frac{\left(-\frac{1}{2}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{20}\vartheta_{n-\frac{1}{4}} + \Lambda_{21}\vartheta_{n-\frac{1}{2}} + \Lambda_{22}\vartheta_n + \Lambda_{23}\vartheta_{n+\frac{1}{4}} + \Lambda_{24}\vartheta_{n+\frac{1}{2}} + \Lambda_{25}\vartheta_{n+\frac{3}{4}} + \Lambda_{26}\vartheta_{n+1}\right) \\ L[\rho(\eta_n);h] - \rho_{n+\frac{1}{4}} - \rho_n - \frac{1}{4}h\rho'_n - \frac{\left(\frac{1}{4}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{30}\vartheta_{n-\frac{1}{4}} + \Lambda_{31}\vartheta_{n-\frac{1}{2}} + \Lambda_{32}\vartheta_n + \Lambda_{33}\vartheta_{n+\frac{1}{4}} + \Lambda_{34}\vartheta_{n+\frac{1}{2}} + \Lambda_{35}\vartheta_{n+\frac{3}{4}} + \Lambda_{36}\vartheta_{n+1}\right) \\ L[\rho(\eta_n);h] - \rho_{n+\frac{1}{2}} - \rho_n - \frac{1}{2}h\rho'_n - \frac{\left(\frac{1}{2}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{40}\vartheta_{n-\frac{1}{4}} + \Lambda_{41}\vartheta_{n-\frac{1}{2}} + \Lambda_{42}\vartheta_n + \Lambda_{43}\vartheta_{n+\frac{1}{4}} + \Lambda_{44}\vartheta_{n+\frac{1}{2}} + \Lambda_{45}\vartheta_{n+\frac{3}{4}} + \Lambda_{46}\vartheta_{n+1}\right) \\ L[\rho(\eta_n);h] - \rho_{n+\frac{3}{4}} - \rho_n - \frac{3}{4}h\rho'_n - \frac{\left(\frac{3}{4}h\right)^2}{2!}\rho''_n - h^3\left(\Lambda_{50}\vartheta_{n-\frac{1}{4}} + \Lambda_{51}\vartheta_{n-\frac{1}{2}} + \Lambda_{52}\vartheta_n + \Lambda_{53}\vartheta_{n+\frac{1}{4}} + \Lambda_{54}\vartheta_{n+\frac{1}{2}} + \Lambda_{55}\vartheta_{n+\frac{3}{4}} + \Lambda_{56}\vartheta_{n+1}\right) \\ L[\rho(\eta_n);h] - \rho_{n+1} - \rho_n - h\rho'_n - \frac{(h)^2}{2!}\rho''_n - h^3\left(\Lambda_{60}\vartheta_{n-\frac{1}{4}} + \Lambda_{61}\vartheta_{n-\frac{1}{2}} + \Lambda_{62}\vartheta_n + \Lambda_{63}\vartheta_{n+\frac{1}{4}} + \Lambda_{64}\vartheta_{n+\frac{1}{2}} + \Lambda_{65}\vartheta_{n+\frac{3}{4}} + \Lambda_{66}\vartheta_{n+1}\right) \end{array}\right\} \quad (3.1)$$

$$\left.\begin{aligned}
L[\rho'(\rho_n);h] &= \rho'_{n-\frac{1}{4}} - \rho'_n + \frac{1}{4}h\rho''_n - h^2 f\left(X_{110}\vartheta_{n-\frac{1}{4}} + X_{111}\vartheta_{n-\frac{1}{2}} + X_{112}\vartheta_n + X_{113}\vartheta_{n+\frac{1}{4}} + X_{114}\vartheta_{n+\frac{1}{2}} + X_{115}\vartheta_{n+\frac{3}{4}} + X_{116}\vartheta_{n+1}\right)\\
L[\rho'(\rho_n);h] &= \rho'_{n-\frac{1}{2}} - \rho'_n + \frac{1}{2}h\rho''_n - h^2\left(X_{120}\vartheta_{n-\frac{1}{4}} + X_{121}\vartheta_{n-\frac{1}{2}} + X_{122}\vartheta_n + X_{123}\vartheta_{n+\frac{1}{4}} + X_{124}\vartheta_{n+\frac{1}{2}} + X_{125}\vartheta_{n+\frac{3}{4}} + X_{126}\vartheta_{n+1}\right)\\
L[\rho'(\rho_n);h] &= \rho'_{n+\frac{1}{4}} - \rho'_n - \frac{1}{4}h\rho''_n - h^2\left(X_{130}\vartheta_{n-\frac{1}{4}} + X_{131}\vartheta_{n-\frac{1}{2}} + X_{132}\vartheta_n + X_{133}\vartheta_{n+\frac{1}{4}} + X_{134}\vartheta_{n+\frac{1}{2}} + X_{135}\vartheta_{n+\frac{3}{4}} + X_{136}\vartheta_{n+1}\right)\\
L[\rho'(\rho_n);h] &= \rho'_{n+\frac{1}{2}} - \rho'_n - \frac{1}{2}h\rho''_n - h^2\left(X_{140}\vartheta_{n-\frac{1}{4}} + X_{141}\vartheta_{n-\frac{1}{2}} + X_{142}\vartheta_n + X_{143}\vartheta_{n+\frac{1}{4}} + X_{144}\vartheta_{n+\frac{1}{2}} + X_{145}\vartheta_{n+\frac{3}{4}} + X_{146}\vartheta_{n+1}\right)\\
L[\rho'(\rho_n);h] &= \rho'_{n+\frac{3}{4}} - \rho'_n - \frac{3}{4}h\rho''_n - h^2\left(X_{150}\vartheta_{n-\frac{1}{4}} + X_{151}\vartheta_{n-\frac{1}{2}} + X_{152}\vartheta_n + X_{153}\vartheta_{n+\frac{1}{4}} + X_{154}\vartheta_{n+\frac{1}{2}} + X_{155}\vartheta_{n+\frac{3}{4}} + X_{156}\vartheta_{n+1}\right)\\
L[\rho'(\rho_n);h] &= \rho'_{n+1} - \rho'_n - h\rho''_n - h^2\left(X_{160}\vartheta_{n-\frac{1}{4}} + X_{161}\vartheta_{n-\frac{1}{2}} + X_{162}\vartheta_n + X_{163}\vartheta_{n+\frac{1}{4}} + X_{164}\vartheta_{n+\frac{1}{2}} + X_{165}\vartheta_{n+\frac{3}{4}} + X_{166}\vartheta_{n+1}\right)
\end{aligned}\right\} \quad (3.2)$$

$$\left.\begin{aligned}
L[\rho''(\eta_n);h] &= \rho''_{n-\frac{1}{4}} - \rho''_n - h\left(X_{210}\vartheta_{n-\frac{1}{4}} + X_{211}\vartheta_{n-\frac{1}{2}} + X_{212}\vartheta_n + X_{213}\vartheta_{n+\frac{1}{4}} + X_{214}\vartheta_{n+\frac{1}{2}} + X_{215}\vartheta_{n+\frac{3}{4}} + X_{216}\vartheta_{n+1}\right)\\
L[\rho''(\eta_n);h] &= \rho''_{n-\frac{1}{2}} - \rho''_n - h\left(X_{220}\vartheta_{n-\frac{1}{4}} + X_{221}\vartheta_{n-\frac{1}{2}} + X_{222}\vartheta_n + X_{223}\vartheta_{n+\frac{1}{4}} + X_{224}\vartheta_{n+\frac{1}{2}} + X_{225}\vartheta_{n+\frac{3}{4}} + X_{226}\vartheta_{n+1}\right)\\
L[\rho''(\eta_n);h] &= \rho''_{n+\frac{1}{4}} - \rho''_n - h\left(X_{230}\vartheta_{n-\frac{1}{4}} + X_{231}\vartheta_{n-\frac{1}{2}} + X_{232}\vartheta_n + X_{233}\vartheta_{n+\frac{1}{4}} + X_{234}\vartheta_{n+\frac{1}{2}} + X_{235}\vartheta_{n+\frac{3}{4}} + X_{236}\vartheta_{n+1}\right)\\
L[\rho''(\eta_n);h] &= \rho''_{n+\frac{1}{2}} - \rho''_n - h\left(X_{240}\vartheta_{n-\frac{1}{4}} + X_{241}\vartheta_{n-\frac{1}{2}} + X_{242}\vartheta_n + X_{243}\vartheta_{n+\frac{1}{4}} + X_{244}\vartheta_{n+\frac{1}{2}} + X_{245}\vartheta_{n+\frac{3}{4}} + X_{246}\vartheta_{n+1} + \right)\\
L[\rho''(\eta_n);h] &= \rho''_{n+\frac{3}{4}} - \rho''_n - h\left(X_{250}\vartheta_{n-\frac{1}{4}} + X_{251}\vartheta_{n-\frac{1}{2}} + X_{252}\vartheta_n + X_{253}\vartheta_{n+\frac{1}{4}} + X_{254}\vartheta_{n+\frac{1}{2}} + X_{255}\vartheta_{n+\frac{3}{4}} + X_{256}\vartheta_{n+1} + \right)\\
L[\rho''(\eta_n);h] &= \rho''_{n+1} - \rho''_n - h\left(X_{260}\vartheta_{n-\frac{1}{4}} + X_{261}\vartheta_{n-\frac{1}{2}} + X_{262}\vartheta_n + X_{263}\vartheta_{n+\frac{1}{4}} + X_{264}\vartheta_{n+\frac{1}{2}} + X_{265}\vartheta_{n+\frac{3}{4}} + X_{266}\vartheta_{n+1}\right)
\end{aligned}\right\} \quad (3.3)$$

ii. The local truncation error of BHM assume $\rho(\eta)$ to be sufficiently differentiable and expanding equation (3.1) to (3.3) about $\eta_n$ using Taylor series to have

$$L_{-\frac{1}{4}}[\rho(\eta_n);h] = (3.5374\times10^{-10}),\ L_{-\frac{1}{2}}[\rho(\eta_n);h] = (1.8165\times10^{-09}),\ L_{\frac{1}{4}}[\rho(\eta_n);h] = (2.3495\times10^{-10})$$
$$L_{\frac{1}{2}}[\rho(\eta_n);h] = (1.5474\times10^{-09}),\ L_{\frac{3}{4}}[\rho(\eta_n);h] = (3.4486\times10^{-09}),\ L_1[\rho(\eta_n);h] = (6.4587\times10^{-09}),$$
$$L_{-\frac{1}{4}}[\rho'(\eta_n);h] = (-5.0501\times10^{-09}),\ L_{-\frac{1}{2}}[\rho'(\eta_n);h] = (1.8838\times10^{-09}),\ L_{\frac{1}{4}}[\rho'(\eta_n);h] = (3.0948\times10^{-09})$$
$$L_{\frac{1}{2}}[\rho'(\eta_n);h] = (6.1896\times10^{-09}),\ L_{\frac{3}{4}}[\rho'(\eta_n);h] = (1.1240\times10^{-09}),\ L_1[\rho'(\eta_n);h] = (4.3058\times10^{-09}),$$
$$L_{-\frac{1}{4}}[\rho''(\eta_n);h] = (4.4278\times10^{-08}),\ L_{-\frac{1}{2}}[\rho''(\eta_n);h] = (-1.2917\times10^{-07}),\ L_{\frac{1}{4}}[\rho''(\eta_n);h] = (2.4094\times10^{-08})$$
$$L_{\frac{1}{2}}[\rho''(\eta_n);h] = (-7.7370\times10^{-10}),\ L_{\frac{3}{4}}[\rho''(\eta_n);h] = (4.278\times10^{-08}),\ L_1[\rho''(\eta_n);h] = (-1.2917\times10^{-09}),$$

**Proof**

Expanding equations (3.1) to (3.3) using Corollary 3.2 and subsequently grouping together terms with similar powers yields the following expression:

$$L_{-\frac{1}{4}}[\rho(\eta_n);h] = (3.5374\times10^{-10})C_{07}h^{07}y^{07}(t_n) + 0(h^{10}),\ L_{-\frac{1}{2}}[\rho(\eta_n);h] = (1.8165\times10^{-09})C_{07}h^{07}y^{07}(t_n) + 0(h^{10}),$$
$$L_{\frac{1}{4}}[\rho(\eta_n);h] = (2.3495\times10^{-10})C_{07}h^{07}y^{07}(t_n) + 0(h^{10}),\ L_{\frac{1}{2}}[\rho(\eta_n);h] = (1.5474\times10^{-09})C_{07}h^{07}y^{07}(t_n) + 0(h^{10}),$$
$$L_{\frac{3}{4}}[\rho(\eta_n);h] = (3.4486\times10^{-09})C_{07}h^{07}y^{07}(t_n) + 0(h^{10}),\ L_1[\rho(\eta_n);h] = (6.4587\times10^{-09})C_{07}h^{07}y^{07}(t_n) + 0(h^{10}),$$
$$L_{-\frac{1}{4}}[\rho'(\eta_n);h] = (-5.0501\times10^{-09})C_{07}h^{07}y^{07}(t_n) + 0(h^{09}),\ L_{-\frac{1}{2}}[\rho'(\eta_n);h] = (1.8838\times10^{-09})C_{07}h^{07}y^{07}(t_n) + 0(h^{09}),$$
$$L_{\frac{1}{4}}[\rho'(\eta_n);h] = (3.0948\times10^{-09})C_{07}h^{07}y^{07}(t_n) + 0(h^{09}),\ L_{\frac{1}{2}}[\rho'(\eta_n);h] = (6.1896\times10^{-09})C_{07}h^{07}y^{07}(t_n) + 0(h^{09}),$$
$$L_{\frac{3}{4}}[\rho'(\eta_n);h] = (1.1240\times10^{-09})C_{07}h^{07}y^{07}(t_n) + 0(h^{09}),\ L_1[\rho'(\eta_n);h] = (4.3058\times10^{-09})C_{07}h^{07}y^{07}(t_n) + 0(h^{09}),$$

$$L_{-\frac{1}{4}}[\rho''(\eta_n);h]=\left(4.4278\times10^{-08}\right)C_{07}h^{07}y^{07}(t_n)+0\left(h^{08}\right),\ L_{-\frac{1}{2}}[\rho''(\eta_n);h]=\left(-1.2917\times10^{-07}\right)C_{07}h^{07}y^{07}(t_n)+0\left(h^{08}\right),$$
$$L_{\frac{1}{4}}[\rho''(\eta_n);h]=\left(2.4094\times10^{-08}\right)C_{07}h^{07}y^{07}(t_n)+0\left(h^{08}\right),\ L_{\frac{1}{2}}[\rho''(\eta_n);h]=\left(-7.7370\times10^{-10}\right)C_{07}h^{07}y^{07}(t_n)+0\left(h^{08}\right),$$
$$L_{\frac{3}{4}}[\rho''(\eta_n);h]=\left(4.278\times10^{-08}\right)C_{07}h^{07}y^{07}(t_n)+0\left(h^{08}\right),\ L_{1}[\rho''(\eta_n);h]=\left(-1.2917\times10^{-09}\right)C_{07}h^{07}y^{07}(t_n)+0\left(h^{08}\right)$$

**Consistency of BHM**

According to Sabo and Adeyeye (2025) the BHM convolves when it's either higher than zero or equals zero. Therefore, this stipulation fits with the BHM derived in this study, and the antinomy truly is consistent.

**Zero Stability of BHM**

A BHM is said to be Zero-stable for any well behaved initial value problem provided if

i. all roots of $\rho(r)$ lies in the unit disk, $|r|\leq 1$

ii. any roots on the unit circle $(|r|\leq 1)$ are simple (Sabo & Adeyeye, 2025)

Hence

$$\rho(z)=\frac{91674240z-1068480z^2+4689496z^3+46746z^4+26397z^5}{+58060800-14515200z+604800z^2+151200z^3-15120z^4-630z^5 135z^6} \quad (3.4)$$

Now set (3.4) equal to zero and solving for $z$ gives $z=1$, hence the method is zero stable.

**Convergence of BHM**

The necessary condition for BHM convergence requires both consistency and zero-stability requirements to be met according to Sabo and Adeyeye (2025), Sunday *et al*. (2025). The BHM system demonstrates both essential requirements so the system can be proven to converge.

**Region of Absolute Stability of BHM**

**Definition 3.3**

The region of absolute stability of BHM is the set of complex values $\lambda h$ for which all solutions of the test problem $y'''=-\lambda^3 y$ will remain bounded as $n\to\infty$.

The concept of A-stability according to (Sabo & Adeyeye, 2025) is discussed by applying the test equation

$$y^{(k)}=\lambda^{(k)}y \quad (3.5)$$

to yield

$$Y_m=\lambda(x)Y_{m-1},\ x=\lambda h \quad (3.6)$$

where $\lambda(x)$ is the amplification matrix of the form

$$\lambda(x)=\left(\upsilon^0-xt^{(0)}-x^4t^{(0)}\right)^{-1}\left(\xi^1-xt^{(1)}-x^4t^{(1)}\right)$$
(3.7)

The matrix $\mu(z)$ has Eigen values $(0,0,\cdots,\nu_k)$ where $\nu_k$ is called the stability function.

The boundary locus method is adopted in generating the stability polynomial of the hybrid method. The polynomial is

$$\bar{h}(w)=\left(-\frac{1}{430080}w^5+\frac{1}{430080}w^6\right)h^6+\left(-\frac{1}{92160}w^6-\frac{1}{92160}w^5\right)h^5+\left(\frac{1}{3840}w^5-\frac{1}{3840}w^6\right)h^4+\left(\frac{1}{384}w^6+\frac{1}{384}w^5\right)h^3$$
$$+\left(-\frac{1}{96}w^7+\frac{1}{96}w^8\right)h^2+\left(-\frac{1}{4}w^6-\frac{1}{4}w^5\right)h-w^5+w^6 \quad (3.8)$$

The polynomial (3.8) is used to plot the region as shown in Fig. 1.

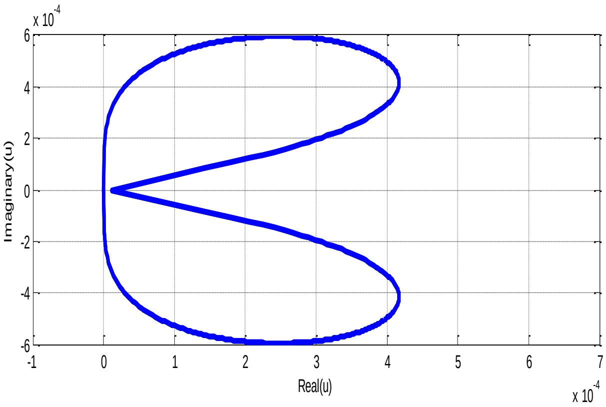


**Figure 1: A-stable regions of absolute stability of BHM**

**Numerical Integration of Block Hybrid Method (BHM)**

The New Block Algorithm (BHM) was numerically applied on Volterra integro-differential equation of second kind. The numerical results are compared with the existing once in literature, textually showing the graphical curve and in a tabular form.

**Example 4.1**

The Volterra integro-differential equation of the form

$$\rho'(\xi)=-\int_0^{\xi}\rho(\tau)d\tau,\ \rho(0)=0,\ 0\leq\xi\leq 1 \quad (4.1)$$

is consider with exact solution given by;

$$\rho(\xi)=\cos(\xi) \quad (4.2)$$

**Source:** Faires and Burden (2015), Majid and Mohamed (2019), Olowe *et al.* (2023).

**Example 4.2**
The Volterra integro-differential equation of the form

$$\rho'(\xi) = \xi \exp(1-\rho(\xi)) - \frac{1}{(1+\xi)^2} - \xi - \int_0^{\xi} \frac{1}{(1+\tau)^2} \exp(1-\rho(\xi))d\tau, \rho(0) = 1 \qquad (4.3)$$

is consider with exact solution given by

$$\rho(\xi) = \frac{1}{1+\xi} \qquad (4.4)$$

**Source:** Faires and Burden (2015), Majid and Mohamed (2019).

**Example 4.3**
The nonlinear Volterra integro-differential equation of the form

$$\rho''(\xi) + \int_0^{\xi} (\rho(\tau))^2 d\tau + \left(\frac{\xi}{2} - \sinh(\xi) - \frac{1}{4}\sinh(2\xi)\right) = 0, \rho(0) = 0, \rho'(0) = 1 \qquad (4.5)$$

is consider with exact solution given by

$$\rho(\xi) = \sinh(\xi) \qquad (4.6)$$

**Source:** Kamoh *et al.* (2017).

The following symbol/acronyms were used in the tables, figures, discussion of results.

| Acronyms | Meaning |
|---|---|
| $\xi$ | Points of Evaluation |
| EABM5 | Error in Fifth Order Adams-Bashforth-Moulton Predictor-Corrector Method of Faires and Burden (2015) |
| E2P3B | Error in Two Point Three-Step Block Method as in This Research of Majid and Mohamed (2019) |
| ECMM | Error in Continuous multistep method of Kamoh *et al.* (2017) |
| E4SBM | Error in Four Step Block Method Developed by Kamoh *et al.* (2017) |
| E5SBM | Error in Five Step Block Method Developed by Kamoh *et al.* (2017) |

**Discussion of Results**
The derivation of the Block Hybrid Method (BHM) is based on the framework of the Linear Block Algorithm (LBA) for solving Volterra integro-differential equations of the second kind. The development begins with a linear multistep formulation, from which a distinct numerical scheme for the BHM is constructed. By sequentially solving the governing block relations along with their higher-order derivatives, a polynomial approximation is developed to represent the solution function. The generalized algorithm is then expanded and simplified to determine the unknown coefficients of the polynomial and its derivatives, leading to the final computational scheme of the BHM. This systematic derivation preserves consistency with established block method procedures while enhancing computational efficiency.
The basic properties of the BHM were also examined to establish its reliability and mathematical soundness. The local truncation error was analyzed using linear difference operators and Taylor series expansion, which showed that the scheme satisfies the required accuracy conditions. The method was further shown to be consistent since its order of convergence is nonnegative. Zero-stability was verified by examining the roots of the characteristic polynomial, where all roots were found to lie within the unit disk and any roots on the unit circle were simple, confirming the stability of the scheme. Since the BHM satisfies both consistency and zero-stability conditions, the method is therefore convergent. Additionally, the region of absolute stability was determined using the boundary locus approach applied to the test equation, and the resulting stability polynomial demonstrated that the BHM possesses A-stable characteristics, indicating bounded solution behavior for suitable complex step parameters.
The numerical results as presented in Tables 1 to 3 illustrate the performance of the (BHM) when applied to Volterra integro-differential equations of the second kind. Similarly, the graphical curves of the Tables are shown in Figs 1-3 accordingly. In Example 4.1 (Table 1), the numerical results produced by the BHM are in complete agreement with the analytical results across all selected points of evaluation. As the step size decreases progressively, the computed solutions follow the exact solution consistently, indicating strong convergence behavior. When compared with the Fifth Order Adams–Bashforth–Moulton method and the Two-Point Three-Step Block Method, the BHM demonstrates closer agreement with the analytical values throughout the interval of integration.

**Table 1: Numerical results of Example 4.1**

| $\xi$ | Analytical Result | Numerical Result | EBHM | EABM5 | E2P3B |
|---|---|---|---|---|---|
| 0.025 | 0.02499739591471233066 | 0.02499739591471233066 | 0.0000e00 | 2.8951e-07 | 5.7323e-08 |
| 0.0125 | 0.01249967448170978872 | 0.01249967448170978872 | 0.0000e00 | 3.6127e-08 | 5.5893e-09 |
| 0.00625 | 0.00624995930997530612 | 0.00624995930997530612 | 0.0000e00 | 4.3953e-09 | 2.2443e-10 |
| 0.003125 | 0.00312499491373946269 | 0.00312499491373946269 | 0.0000e00 | 5.4213e-10 | 1.3908e-11 |
| 0.0015625 | 0.00156249936421720001 | 0.00156249936421720001 | 0.0000e00 | 6.7325e-11 | 8.6930e-13 |
| 0.00078125 | 0.00078124992052714272 | 0.00078124992052714272 | 0.0000e00 | 8.3688e-12 | 5.4179e-14 |

**Table 2: Numerical results of Example 4.2**

| $\xi$ | Analytical Result | Numerical Result | EBHM | EABM5 | E2P3B |
|---|---|---|---|---|---|
| 0.025 | 0.97560975609756097561 | 0.97560975609921673678 | 1.6558e-12 | 1.7212e-08 | 8.3237e-08 |
| 0.0125 | 0.98765432098765432099 | 0.98765432098893626373 | 1.2819e-12 | 2.0551e-09 | 3.8384e-09 |
| 0.00625 | 0.99378881987577639752 | 0.99378881987578963537 | 1.3238e-14 | 1.9089e-10 | 2.0775e-10 |
| 0.003125 | 0.99688473520249221184 | 0.99688473520249809256 | 5.8807e-15 | 1.1926e-11 | 1.2654e-11 |
| 0.0015625 | 0.99843993759750390016 | 0.99843993759750339824 | 5.0192e-16 | 7.4529e-13 | 9.6889e-13 |
| 0.00078125 | 0.99921935987509758002 | 0.99921935987509772187 | 1.4185e-16 | 4.6518e-14 | 4.3676e-13 |

**Table 3: Numerical results of Example 4.3**

| $\xi$ | Analytical Result | Numerical Result | EBHM | ECMM | E4SBM | E5SBM |
|---|---|---|---|---|---|---|
| 0.16 | 0.16068354101279944828 | 0.16068354101279129786 | 8.1504e-15 | 4.5700e-08 | 4.5800e-08 | 4.5700e-08 |
| 0.32 | 0.32548936363113307984 | 0.32548936363113985654 | 6.7767e-15 | 3.7140e-07 | 3.7160e-07 | 3.7150e-07 |
| 0.48 | 0.49864550519337626463 | 0.49864550519337347295 | 2.7917e-15 | 1.2858e-06 | 1.2861e-06 | 1.2859e-06 |
| 0.64 | 0.68459422763095139805 | 0.68459422763099268709 | 4.1289e-14 | 3.1554e-06 | 3.1559e-06 | 3.1555e-06 |
| 0.80 | 0.88810598218762300658 | 0.88810598218761237564 | 1.0631e-14 | 6.4379e-06 | 6.4386e-06 | 6.4381e-06 |
| 0.96 | 1.11440179372400284780 | 1.11440179374552187898 | 2.1519e-11 | 1.1717e-05 | 1.1718e-05 | 1.1718e-05 |

In Table 2 (Example 4.2), the BHM continues to show excellent agreement with the exact solution at all evaluation points. The computed values approach the analytical solution more closely as the mesh is refined, confirming the stability and consistency of the method. Although the existing methods in the literature also produce satisfactory approximations, the BHM results align more precisely with the analytical values at each step size. This consistent closeness suggests that the proposed method possesses strong accuracy and reliable convergence characteristics for linear Volterra integro-differential equations.

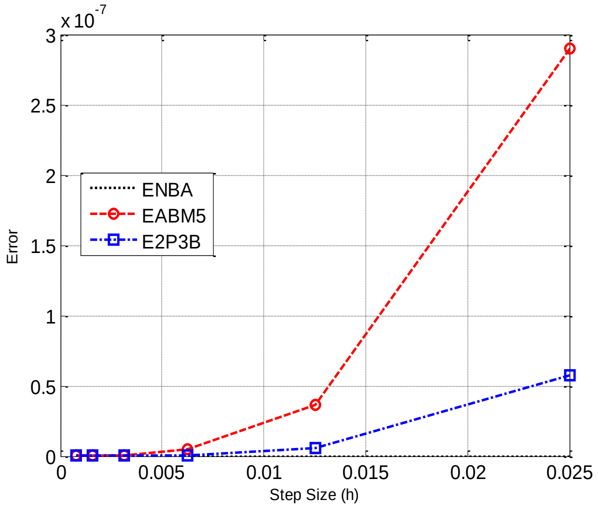


**Figure 1: Graphical curve of Table 1**

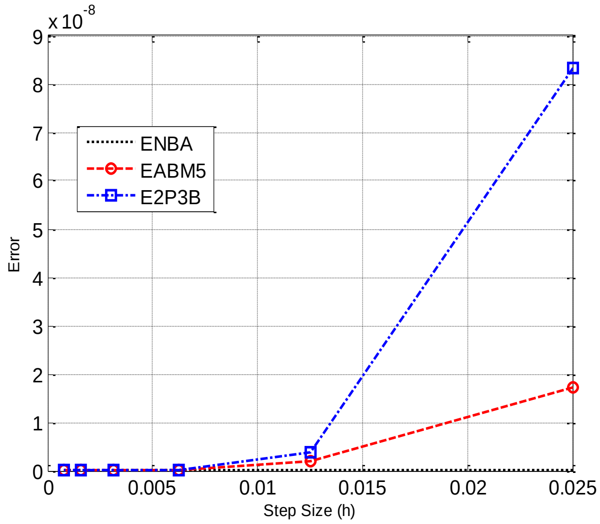


**Figure 2: Graphical curve of Table 2**

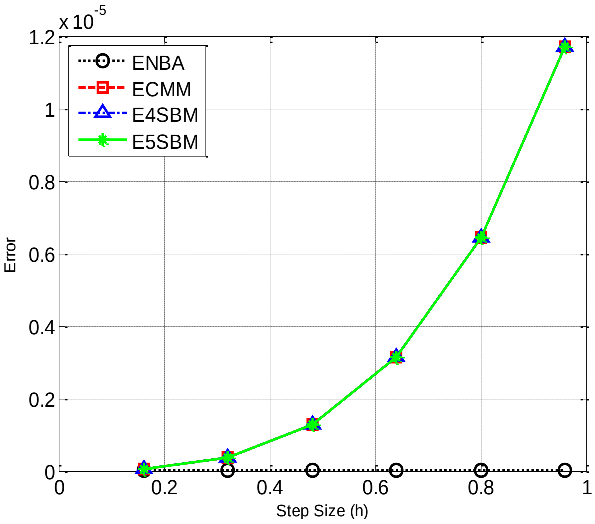


**Figure 3: Graphical curve of Table 3**

For the nonlinear case presented in Table 3 (Example 4.3), the effectiveness of the BHM is further confirmed. The numerical solutions generated by the method closely track the analytical solution across the entire domain. Even as the solution values increase with the points of evaluation, the BHM maintains remarkable agreement with the exact results. In comparison with the Continuous Multistep Method and the Four-Step and Five-Step Block Methods, the BHM exhibits better alignment with the analytical solution, demonstrating its capability to handle nonlinear problems efficiently without loss of stability.

## SUMMARY AND CONCLUSION

This study presented the development and analysis of a Block Hybrid Method (BHM) for the numerical solution of Volterra integro-differential equations of the second kind. The proposed BHM was formulated within the framework of the Linear Block Algorithm, where polynomial approximations were constructed through the sequential evaluation of block relations and their higher-order derivatives. Fundamental properties of the method, including local truncation error, consistency, zero-stability, convergence, and region of absolute stability, were rigorously investigated. The order of the method was established using Taylor series expansion and linear difference operator techniques, while the zero-stability property was verified through the root condition of the characteristic polynomial.

To assess the performance of the proposed scheme, numerical experiments were conducted on three benchmark test problems comprising both linear and nonlinear Volterra integro-differential equations. The numerical results were compared with those obtained using existing methods, including the Adams–Bashforth–Moulton predictor–corrector method, continuous multistep methods, and other related block schemes. The outcomes demonstrated that the proposed BHM generated numerical solutions that were in close agreement with the exact solutions for various step sizes, thereby confirming the accuracy and reliability of the method.

Overall, the developed Block Hybrid Method has proven to be an efficient, accurate, and robust computational technique for solving Volterra integro-differential equations of the second kind. The method exhibited strong convergence characteristics, improved numerical stability, and superior error minimization when compared with several existing numerical approaches. Consequently, the proposed scheme provides a reliable alternative for the numerical treatment of integro-differential equations arising in scientific and engineering applications where analytical solutions are difficult or impossible to obtain. Future research may consider extending the proposed algorithm to fractional-order integro-differential equations, delay integro-differential systems, and higher-dimensional problems.

**Conflict of interest:** The authors declare no conflict of interest.

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