

A GRAPH-THEORETIC STUDY OF TRANSFORMATIONS WITH ORDER-PRESERVING AND CONTRACTION RESTRICTIONS ON A FINITE CHAIN

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ABSTRACT

Every transformation $\alpha \in T_n$ naturally defines a directed graph, called the functional digraph of α , whose vertex set is $X_n = \{1, 2, \dots, n\}$ and whose edge set is $E\alpha = \{(x, \alpha(x)) : x \in X_n\}$. Using this representation, we investigate a class of transformations whose associated digraphs satisfy different structural conditions on complementary subsets of the vertex set. For a fixed subset $Y \subseteq X_n$, we consider the set $K = \{\alpha \in T_n : \alpha|_Y \text{ is Order-Preserving and } \alpha|_{X_n \setminus Y} \text{ is a contraction}\}$. From a graph-theoretic perspective, this corresponds to functional digraphs whose induced subdigraph on Y preserves the chain's natural order, while the induced subdigraph on $X_n \setminus Y$ satisfies a distance-nonincreasing condition. We show that a transformation belongs to K if and only if these graph properties hold on the corresponding induced subdigraphs. Furthermore, the functional digraph of each element of K admits a natural decomposition into two components: an Order-Preserving digraph on Y and a contraction digraph on $X_n \setminus Y$. We prove that the set K is closed under composition and therefore forms a subsemigroup of T_n when viewed through the composition of their associated functional digraphs. In addition, we characterize the idempotent elements of K in graph-theoretic terms by showing that α is idempotent if and only if every vertex in $\Gamma(\alpha)$ maps to a fixed point after one step. These results provide a graph-theoretic framework for studying hybrid classes of transformations on finite chains and contribute to the understanding of structural properties of functional digraphs arising from constrained transformations.

Keywords: Order preserving; Contraction; Idempotent

INTRODUCTION

Transformation semigroups admit natural interpretations in terms of directed graphs, which makes graph-theoretic methods a useful tool for studying their structural properties. In particular, every transformation $\alpha : X_n \rightarrow X_n$ on a finite set $X_n = \{1, 2, \dots, n\}$ determines a directed graph, commonly called the functional digraph of α , whose vertex set is X_n and in which each vertex x has a directed edge to $\alpha(x)$. This graphical representation provides a convenient way of interpreting algebraic properties of transformations in terms of structural properties of directed graphs. Consequently, transformation semigroups have been investigated using graph-theoretic techniques in several recent works (Garba *et al.*, 2017; Eze *et al.*, 2025; Eze *et al.*, 2025). Among the most fundamental objects in this context is the full transformation semigroup T_n , consisting of all mappings from X_n into itself under composition (Howie, 1995). The elements of T_n can be viewed as functional digraphs on X_n , and many properties of these transformations can therefore be studied through the structure of their associated graphs. Standard references on the theory of transformation semigroups include the works of Howie (1995) and Ganyushkin and Mazorchuk (2008), where various structural and combinatorial aspects of transformation semigroups are discussed. Several studies have focused on classes of transformations defined by imposing structural restrictions on their action (Gould *et al.*, 2023; Zubairu *et al.*, 2022). One important example is the class of order-

preserving transformations on a finite chain. A transformation α on X_n is said to be order-preserving if $x \leq y$ implies $\alpha(x) \leq \alpha(y)$ for all $x, y \in X_n$. From a graph-theoretic point of view, such transformations correspond to functional digraphs whose edges respect the natural ordering of vertices in the chain. These transformations have attracted considerable attention due to their interesting structural and combinatorial properties (Zhang & Luo, 2017), and various aspects of their associated digraphs have been investigated (Sommanee & Sanwong, 2025).

Another important class of transformations consists of contractions on finite chains. A transformation α is called a contraction if it does not increase the distance between elements, that is, $|\alpha(x) - \alpha(y)| \leq |x - y|$ for all $x, y \in X_n$. In graph-theoretic terms, the functional digraph of a contraction satisfies a distance non-increasing condition with respect to the natural metric on the chain. Such transformations have been studied in connection with the combinatorial and structural properties of their associated directed graphs (Zubairu *et al.*, 2022; Zhao & Hu, 2023).

Motivated by these developments, it is natural to consider transformations whose associated digraphs satisfy different structural conditions on different parts of the vertex set. Such hybrid constructions often lead to directed graphs with interesting structural features that combine properties arising from distinct classes of transformations. In this paper, we study a class of transformations whose functional digraphs exhibit two

different types of structural behavior on complementary subsets of the underlying chain.

More precisely, for a subset $Y \subseteq X_n$, we consider transformations whose restriction to Y is Order-Preserving and whose restriction to the complement $X_n \setminus Y$ is a contraction. Equivalently, the associated functional digraph consists of a substructure on Y that respects the natural ordering of vertices and a complementary substructure on $X_n \setminus Y$ satisfying a distance non-increasing condition.

The aim of this paper is to investigate the structural properties of these transformations from a graph-theoretic perspective. In particular, we analyze the structure of the associated functional digraphs, describe their decomposition into Order-Preserving and contraction components, and examine structural features such as fixed points and image structures within the corresponding graphs.

The results obtained in this work contribute to the graph-theoretic study of transformation semigroups and provide a framework for further investigations of directed graph structures arising from constrained transformations on finite chains.

Preliminaries

Let $X_n = \{1, 2, \dots, n\}$ be a finite chain with the usual order.

Definition 2.1: The full transformation semigroup on X_n is $T_n = \{\alpha : X_n \rightarrow X_n\}$ under the operation of composition of mappings.

Definition 2.2: A transformation $\alpha \in T_n$ is said to be Order-Preserving if $x \leq y \Rightarrow \alpha(x) \leq \alpha(y)$ for all $x, y \in X_n$.

Definition 2.3: A transformation $\alpha \in T_n$ is called a contraction if $|\alpha(x) - \alpha(y)| \leq |x - y|$ for all $x, y \in X_n$.

Let $Y \subseteq X_n$ and denote its complement by $X_n \setminus Y$.

Definition 2.4: Define $K = \{\alpha \in T_n : \alpha|_Y \text{ is Order-Preserving and } \alpha|_{X_n \setminus Y} \text{ is a contraction}\}$. Thus, elements of K preserve order on Y and act as contractions on the complement of Y .

To study these transformations from a graph-theoretic perspective, we associate a directed graph with each transformation.

Definition 2.5: A directed graph (digraph) is an ordered pair $G = (V, E)$ where V is a nonempty set of vertices and $E \subseteq V \times V$ is a set of directed edges. An edge $(u, v) \in E$ is interpreted as a directed edge from u to v .

Definition 2.6: Let $\alpha \in T_n$. The functional digraph associated with α is the directed graph $\Gamma(\alpha) = (X_n, E_\alpha)$ where $E_\alpha = \{(x, \alpha(x)) : x \in X_n\}$. Thus, every vertex $x \in X_n$ has exactly one outgoing edge directed to $\alpha(x)$.

Definition 2.7: Let $G = (V, E)$ be a digraph and let $U \subseteq V$. The induced subdigraph of G on U is the digraph $G[U] = (U, E_U)$ where $E_U = \{(u, v) \in E : u, v \in U\}$.

Definition 2.8: Let $\alpha \in T_n$. A vertex $x \in X_n$ is called a fixed point of α if $\alpha(x) = x$. In the functional digraph $\Gamma(\alpha)$, a fixed point corresponds to a vertex with a loop.

Definition 2.9: The image of a transformation α is the set $\text{Im}(\alpha) = \{\alpha(x) : x \in X_n\}$. In the functional digraph $\Gamma(\alpha)$, the image corresponds to the set of vertices that receive at least one incoming edge.

MAIN RESULTS

Every transformation $\alpha \in T_n$ naturally defines a directed graph (called the functional digraph of α) $\Gamma(\alpha) = (X_n, E_\alpha)$ where $E_\alpha = \{(x, \alpha(x)) : x \in X_n\}$. Thus, each vertex $x \in X_n$ has exactly one outgoing edge to $\alpha(x)$.

Definition 3.1: Let $Y \subseteq X_n$ and let $\Gamma(\alpha)$ be the digraph associated with $\alpha \in T_n$. We say $\alpha \in K$ if and only if the following graph properties hold:

1. For every edge $(x, \alpha(x))$ with $x \in Y$, the mapping preserves the natural order of vertices, that is, $x \leq y \Rightarrow \alpha(x) \leq \alpha(y)$.
2. For every pair of vertices $x, y \in X_n \setminus Y$, the edges satisfy the contraction property $|\alpha(x) - \alpha(y)| \leq |x - y|$.

Thus, the digraph $\Gamma(\alpha)$ consists of two structural parts:

- an Order-Preserving directed structure on Y ,
- a distance-non-increasing directed structure on $X_n \setminus Y$.

Theorem 3.2: Let $\alpha \in T_n$. Then $\alpha \in K$ if and only if the induced subgraph of $\Gamma(\alpha)$ on Y is Order-Preserving and the induced subgraph on $X_n \setminus Y$ satisfies the contraction condition $|\alpha(x) - \alpha(y)| \leq |x - y|$ for all vertices $x, y \in X_n \setminus Y$.

Proof: Suppose $\alpha \in K$. By definition, the restriction $\alpha|_Y$ is Order-Preserving. Hence for vertices $x, y \in Y$ with $x \leq y$, the edges $(x, \alpha(x))$ and $(y, \alpha(y))$ satisfy $\alpha(x) \leq \alpha(y)$. Therefore, the directed edges in the induced subgraph on Y preserve the linear order.

Similarly, since $\alpha|_{X_n \setminus Y}$ is a contraction, for every $x, y \in X_n \setminus Y$ we have $|\alpha(x) - \alpha(y)| \leq |x - y|$. Thus, the induced subgraph on $X_n \setminus Y$ satisfies the distance contraction property.

Conversely, if the induced subgraphs satisfy these two properties, then the restrictions $\alpha|_Y$ and $\alpha|_{X_n \setminus Y}$ are respectively Order-Preserving and contractions. Hence $\alpha \in K$.

Theorem 3.3: The functional digraph of every element $\alpha \in K$ decomposes as $\Gamma(\alpha) = \Gamma(\alpha|_Y) \cup \Gamma(\alpha|_{X_n \setminus Y})$ where

- $\Gamma(\alpha|_Y)$ is an Order-Preserving directed graph,
- $\Gamma(\alpha|_{X_n \setminus Y})$ is a contraction digraph.

Proof: Let $\alpha \in K$. Define $\alpha_1 = \alpha|_Y$, $\alpha_2 = \alpha|_{X_n \setminus Y}$. Then the edge set of $\Gamma(\alpha)$ can be written as $E_\alpha = \{(x, \alpha_1(x)) : x \in Y\} \cup \{(x, \alpha_2(x)) : x \in X_n \setminus Y\}$. Hence $\Gamma(\alpha) = \Gamma(\alpha_1) \cup \Gamma(\alpha_2)$. Since α_1 is Order-Preserving, $\Gamma(\alpha_1)$ preserves the natural ordering of vertices. Since α_2 is a contraction, $\Gamma(\alpha_2)$ satisfies the metric contraction condition.

Theorem 3.4: The set K forms a subsemigroup of T_n and the associated functional digraphs are closed under composition of transformations.

Proof: Let $\alpha, \beta \in K$ and consider their digraphs $\Gamma(\alpha)$ and $\Gamma(\beta)$. The composition $\alpha\beta$ corresponds to following an edge of β and then an edge of α in the digraph.

For $x, y \in Y$ with $x \leq y$, since β preserves order on Y , $\beta(x) \leq \beta(y)$. Since α also preserves order on Y , $\alpha(\beta(x)) \leq \alpha(\beta(y))$. Hence the composition preserves order on Y .

Similarly, for $x, y \in X_n \setminus Y$, since β is a contraction, $|\beta(x) - \beta(y)| \leq |x - y|$. Applying the contraction property of α gives $|\alpha(\beta(x)) - \alpha(\beta(y))| \leq |\beta(x) - \beta(y)| \leq |x - y|$. Hence the composition also satisfies the contraction property.

Therefore $\alpha\beta \in K$, and the digraphs remain within the same class. Thus, K is a subsemigroup of T_n .

Theorem 3.5: Let $\alpha \in K$. Then α is idempotent if and only if every vertex in the digraph $\Gamma(\alpha)$ maps to a fixed point after one step, that is, $\alpha(\alpha(x)) = \alpha(x)$ for all $x \in X_n$.

Proof: If α is idempotent, then $\alpha^2 = \alpha$. Hence $\alpha(\alpha(x)) = \alpha(x)$ for every vertex x . Thus, every directed path of length two collapses to a loop, meaning that all vertices map to fixed points of the digraph.

Conversely, if the above condition holds, then applying α twice produces the same result as applying it once. Hence $\alpha^2 = \alpha$ and α is idempotent.

Graph-Theoretic Characterization of Fixed Points

Proposition 3.6: Let $\alpha \in K$. A vertex $x \in X_n$ is a fixed point of α if and only if x carries a loop in the functional digraph $\Gamma(\alpha)$. Moreover:

1. Every fixed point in Y is a vertex x such that $(x, x) \in E\alpha$ and the restriction $\alpha|_Y$ preserves the ordering at x .
2. Every fixed point in $X_n \setminus Y$ satisfies $|\alpha(x) - x| = 0$, so the contraction inequality holds with equality zero.

Proof: A fixed point satisfies $\alpha(x) = x$, so the corresponding edge is the loop $(x, x) \in E\alpha$. For $x \in Y$, since $\alpha|_Y$ is Order-Preserving and $\alpha(x) = x$, the edge respects the ordering trivially. For $x \in X_n \setminus Y$, the contraction condition gives $|\alpha(x) - x| = |x - x| = 0 \leq 0$, which is satisfied.

Corollary 3.7: If $\alpha \in K$ is idempotent, then every vertex of $\Gamma(\alpha)$ is either a fixed point or maps in one step to a fixed point. In other words, the functional digraph $\Gamma(\alpha)$ has no directed paths of length greater than one that do not terminate in a loop.

Proof. Since $\alpha^2 = \alpha$, applying α twice to any x gives $\alpha(\alpha(x)) = \alpha(x)$, so $\alpha(x)$ is a fixed point. Hence every non-fixed vertex maps directly to a fixed point after exactly one step.

In-Degree and Image Structure in the Digraph

Theorem 3.8: Let $\alpha \in K$ with functional digraph $\Gamma(\alpha) = (X_n, E\alpha)$. Then the image set $\text{Im}(\alpha)$ coincides exactly with the set of vertices of $\Gamma(\alpha)$ that have in-degree at least one, that is, $\text{Im}(\alpha) = \{v \in X_n : \exists x \in X_n, (x, v) \in E\alpha\}$. Moreover, if $\alpha \in K$ then:

1. The set $\text{Im}(\alpha|_Y) = \alpha(Y)$ is a totally ordered subset of X_n .
2. The restriction $\alpha|_{X_n \setminus Y}$ satisfies $\text{diam}(\text{Im}(\alpha|_{X_n \setminus Y})) \leq \text{diam}(X_n \setminus Y)$, where diam denotes the diameter with respect to the natural metric on X_n .

Proof: The first statement holds for any functional digraph: a vertex v has an incoming edge if and only if $v = \alpha(x)$ for some x , that is, $v \in \text{Im}(\alpha)$.

For part (1): since $\alpha|_Y$ is Order-Preserving and Y is a chain, the image $\alpha(Y)$ is totally ordered in X_n .

For part (2): since $\alpha|_{X_n \setminus Y}$ is a contraction, for any $x, y \in X_n \setminus Y$ we have $|\alpha(x) - \alpha(y)| \leq |x - y| \leq \text{diam}(X_n \setminus Y)$, so the image has diameter at most that of $X_n \setminus Y$.

Green's Relations in Graph-Theoretic Terms

The Green's relations of a transformation semigroup have natural interpretations in terms of the functional digraph.

Theorem 3.9: Let $\alpha, \beta \in K$ with functional digraphs $\Gamma(\alpha)$ and $\Gamma(\beta)$.

1. $\alpha L \beta$ if and only if $\Gamma(\alpha)$ and $\Gamma(\beta)$ have the same set of vertices with in-degree at least one, that is, $\text{Im}(\alpha) = \text{Im}(\beta)$.
2. $\alpha R \beta$ if and only if $\Gamma(\alpha)$ and $\Gamma(\beta)$ induce the same partition of X_n into fibres, that is, the kernel partitions coincide: $\ker(\alpha) = \ker(\beta)$.
3. $\alpha H \beta$ if and only if both the image sets and the kernel partitions coincide.
4. $\alpha J \beta$ if and only if $|\text{Im}(\alpha)| = |\text{Im}(\beta)|$, i.e., the two digraphs have the same number of vertices with positive in-degree.

Proof: Since K is a subsemigroup of T_n , the characterizations of Green's relations in T_n in terms of image and kernel (see [1]) apply. Parts (1) and (2) follow from the general theory. Part (3) follows since $H = L \cap R$. Part (4) follows from the fact that K is a finite semigroup and the J -class is determined by the rank of the transformation, which equals $|\text{Im}(\alpha)|$.

Composition of Functional Digraphs: A Detailed

Example

Example 3.10: Let $X_5 = \{1, 2, 3, 4, 5\}$ and $Y = \{1, 2, 3\}$, so that $X_5 \setminus Y = \{4, 5\}$. Define two transformations $\alpha, \beta \in K$ by

$\alpha =$				
1	2	3	4	5
1	2	3	4	4
$\beta =$				
1	2	3	4	5
2	2	3	4	5

Verification that $\alpha \in K$: On $Y = \{1, 2, 3\}$: $\alpha(1) = 1 \leq 2 = \alpha(2) \leq 3 = \alpha(3)$, so $\alpha|_Y$ is Order-Preserving. On $\{4, 5\}$: $|\alpha(4) - \alpha(5)| = |4 - 4| = 0 \leq 1$, so $\alpha|_{\{4,5\}}$ is a contraction.

Verification that $\beta \in K$: On $Y = \{1, 2, 3\}$: $\beta(1) = 2 \leq 2 = \beta(2) \leq 3 = \beta(3)$, so $\beta|_Y$ is Order-Preserving. On $\{4, 5\}$: $|\beta(4) - \beta(5)| = |4 - 5| = 1 \leq 1$, so $\beta|_{\{4,5\}}$ is a contraction.

Composition $\alpha\beta$:

$\alpha\beta(x) = \alpha(\beta(x))$				
1	2	3	4	5
2	2	3	4	4

On Y : $\alpha\beta(1) = 2 \leq 2 = \alpha\beta(2) \leq 3 = \alpha\beta(3)$, Order-Preserving. On $\{4, 5\}$: $|\alpha\beta(4) - \alpha\beta(5)| = |4 - 4| = 0 \leq 1$, a contraction. Hence $\alpha\beta \in K$.

The functional digraphs of α, β and $\alpha\beta$ illustrate the vertex set $X_5 = \{1, 2, 3, 4, 5\}$, with vertices 1, 2, 3 (belonging to Y) and vertices 4, 5 (belonging to $X_5 \setminus Y$).

Observation: In each digraph, the nodes belonging to Y form an Order-Preserving subdigraph and the nodes

belonging to $X_s \setminus Y$ form a contraction subdigraph. Loops represent fixed points. The composition $\alpha\beta$ inherits both structural properties, confirming that K is closed under composition.

Regularity in Graph-Theoretic Terms

Theorem 3.11: Let $\alpha \in K$. Then α is a regular element of K if and only if there exists $\beta \in K$ such that the functional digraphs satisfy $\Gamma(\alpha\beta\alpha) = \Gamma(\alpha)$. In particular, every idempotent element of K is regular.

Proof: By definition, α is regular if there exists $\beta \in K$ with $\alpha\beta\alpha = \alpha$. Since the functional digraph of a transformation determines it uniquely, $\alpha\beta\alpha = \alpha$ is equivalent to $\Gamma(\alpha\beta\alpha) = \Gamma(\alpha)$.

For the second claim, if α is idempotent then $\alpha^2 = \alpha$, so setting $\beta = \alpha$ gives $\alpha\beta\alpha = \alpha^3 = \alpha^2 \cdot \alpha = \alpha \cdot \alpha = \alpha^2 = \alpha$. Hence α is regular.

Example 3.12: Let $X_s = \{1, 2, 3, 4, 5\}$ and $Y = \{1, 2, 3\}$. Define $\alpha \in K$ as: $\alpha(1) = 2, \alpha(2) = 3, \alpha(3) = 3, \alpha(4) = 4, \alpha(5) = 4$.

This example illustrates the decomposition: $\Gamma(\alpha) = \Gamma(\alpha|_Y) \cup \Gamma(\alpha|_{X_s \setminus Y})$, where:

- The subgraph on $Y = \{1, 2, 3\}$ preserves the natural order (Order-Preserving part),
- The subgraph on $X_s \setminus Y = \{4, 5\}$ satisfies the contraction condition (distance-non-increasing part),
- Loops indicate fixed points.

CONCLUSION

In this paper we investigated a class of transformations on the finite chain X_n from a graph-theoretic perspective by considering the functional digraph associated with each transformation. For a fixed subset $Y \subseteq X_n$, we studied the set $K = \{\alpha \in T_n : \alpha|_Y \text{ is Order-Preserving and } \alpha|_{X_n \setminus Y} \text{ is a contraction}\}$ whose elements are characterized entirely by graph-theoretic properties of their functional digraphs.

Using the functional digraph representation $\Gamma(\alpha) = (X_n, E_\alpha)$, we showed that a transformation belongs to K if and only if the induced subdigraph on Y preserves the natural ordering of vertices while the induced sub-digraph on $X_n \setminus Y$ satisfies a distance non-increasing property. Every such digraph decomposes as $\Gamma(\alpha) = \Gamma(\alpha|_Y) \cup \Gamma(\alpha|_{X_n \setminus Y})$, where the two components carry structurally distinct properties.

We established that:

1. K is a subsemigroup of T_n , and the associated functional digraphs are closed under composition. An explicit composition example with digraph illustrations for $n = 5, Y = \{1, 2, 3\}$ confirmed this closure.
2. Fixed points of any $\alpha \in K$ correspond precisely to looped vertices in $\Gamma(\alpha)$, and idempotents are characterized as those transformations for which every vertex either is a fixed point or maps directly to one after a single step.
3. The image $\text{Im}(\alpha)$ equals the set of vertices of $\Gamma(\alpha)$ with in-degree at least one, and the image of $\alpha|_Y$ is always a totally ordered subset of X_n .
4. Green's L-, R-, H- and J-relations admit complete graph-theoretic descriptions in terms of

coincidence of image sets, kernel partitions, and ranks of the associated digraphs.

5. Every idempotent in K is a regular element, characterized by the existence of $\beta \in K$ satisfying $\Gamma(\alpha\beta\alpha) = \Gamma(\alpha)$.

These results provide a unified graph-theoretic framework for studying hybrid classes of constrained transformations. Several natural directions remain open for further investigation:

1. Determining the rank of K and finding a minimal generating set described in terms of the functional digraph structure.
2. Enumerating the L-, R-, H- and J-classes and characterizing their sizes via digraph parameters.
3. Studying the power digraph of K and investigating periodicity and index properties of the associated transformations.
4. Characterizing the regular elements of K completely and determining conditions under which K forms a regular semigroup.
5. Extending the framework to partial transformations and investigating the resulting functional digraphs of partial hybrid contraction-Order-Preserving maps.

Conflict of interest: The authors declare no conflict of interest.

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REFERENCES

- Eze, C., Imam, A. T., Balarabe, M. and Olaiya, O. O., (2025). Graph-theoretic characterizations of quasi-idempotents in full Order-Preserving transformation semigroup. *Babylonian Journal of Mathematics*. doi:10.58496/bjm/2025/009.
- Eze C., Olaiya O. O. and Kasim, S. (2025). Graph-Theoretic Characterization of Quasi-Nilpotent Elements in Finite Semigroups of Full Order-Preserving Transformations, *Mikailalsys Journal of Mathematics and Statistics*. DOI: 10.58578/mjms.v3i2.5906
- Ganyushkin, O. and Mazorchuk, V. (2008). *Classical Finite Transformation Semigroups: An Introduction*; Vol. 9. Springer Science & Business Media.
- Garba, G. U., Ibrahim, M. J. and Imam, A. (2017). On certain semigroups of full contraction maps of a finite chain. *Turkish Journal of Mathematics*, 41, 500–507. DOI: 10.3906/mat-1602-52
- Gould, V., Grau, A. and Johnson, M. (2023). The Structure of $\text{End}(T_n)$. arXiv.2307.11596[math.RA]

- Howie, J. M. (1995). *Fundamentals of Semigroup Theory*. Oxford University Press, Oxford.
- Sommanee, W. and Sanwong, J. (2015). Order-Preserving transformations with restricted range: regularity, Green's relations, and ideals. *Algebra Universalis*, 74, 277–291. DOI: 10.1007/s00012-015-0354-z
- Zhang, J. and Luo, Y. (2017). On certain Order-Preserving transformation semigroups. *Communications in Algebra*, 45, 2980–2995. DOI: 10.1080/00927872.2016.1233333
- Zhao, P. and Hu, H. (2023). The semigroups of Order-Preserving transformations with restricted range. *Semigroup Forum*, 106, 516–525. DOI: 10.1007/s00233-023-10345-2
- Zubairu, M. M., Umar, A. and Aliyu, M. J. (2022). On the combinatorial and rank properties of certain subsemigroups of full contractions of a finite chain. arXiv.2205.00438v1[math.GR]