

COMPARATIVE ANALYSIS OF METHODS FOR SOLVING SINGULAR INTEGRAL EQUATION OF THE SECOND KIND

C. K. Onuoha*, P. V. Ayoo, and M. A. Mbah

Department of Mathematics, Federal University of Lafia, Nasarawa State, Nigeria

*Corresponding email: unikcaleb@gmail.com

ABSTRACT

Solving singular integral equations (SIEs-2) of the second kind poses a distinct mathematical computation challenge because the unknown function appears both inside and outside the integral. Several numerical methods can be employed to solve this problem but it mostly involves rigorous numerical computations. Methods like the Chebyshev, Bernstein and Legendre polynomial method can be used to solve SIEs-2 but questions like; which method is best suitable, which method is cheaper and more accurate, are left unanswered. This study compares and analyses these three methods, namely: Legendre polynomials, Chebyshev polynomials and Bernstein polynomials. Using the developed methods, three (3) relevant examples were solved and the results were compared for error and accuracy. The analysis highlights the effectiveness of orthogonal polynomial method and the results show that the Chebyshev method gives a better approximation than the Bernstein and Legendre method in terms of convergence and accuracy.

Keywords: Integral equation, Singularity, Chebyshev polynomial, Legendre polynomial, Bernstein polynomial

INTRODUCTION

Singular integral equations (SIEs) are a special class of integral equations characterized mainly by a kernel function that becomes infinite at certain points within the integration domain. These equations arise naturally in the mathematical modeling of physical phenomena across diverse fields, including fluid dynamics, elasticity, fracture mechanics, and quantum mechanics (Saira, 2019). Analytical and numerical methods of solution are made difficult due to the presence of these singularities, requiring specialized techniques to accurately handle the kernel's behavior and the potential singularities in the solution itself. The works of scholars such as Gakhov (2014) and Mikhlin (2014) laid the theoretical groundwork for understanding these equations, while more recent research has focused on developing robust and efficient numerical schemes to solve them.

Legendre polynomials are a family of orthogonal polynomials defined on the interval $[-1, 1]$ with respect to a uniform weight function. They are well-behaved and widely used in numerical analysis due to their simplicity and ease of computation through recurrence relations (Abramowitz & Stegun, 2006). When applied to SIEs, the unknown function is expanded in a series of Legendre polynomials, and collocation points, often the roots of a higher-order Legendre polynomial, are used to transform the integral equation into a linear system. This method is particularly effective for problems where the solution is expected to be relatively smooth across the interval.

Chebyshev polynomials, on the other hand, are renowned for their superior convergence properties in numerical approximation, particularly for functions with endpoint singularities. These polynomials are orthogonal with respect to a non-uniform weight function, which naturally aligns with the typical singular behavior of SIE

solutions on the interval $[-1, 1]$ (Boyd, 2001). The two kinds of Chebyshev polynomials, $T_n(x)$ and $U_n(x)$, offer flexibility. $T_n(x)$ is suitable for solutions unbounded at both endpoints, while $U_n(x)$ is often used for solutions bounded at the endpoints. The use of Chebyshev collocation points (the zeros of the Chebyshev polynomials) is a key feature of this method, as these points minimize the Runge phenomenon and ensure high accuracy.

The Bernstein polynomial method, while less common for direct application to SIEs due to its original definition on the interval $[0, 1]$, can also be adapted to solve these problems through a change of variables. Bernstein polynomials are known for their excellent approximation properties for continuous functions and are the foundation of Bézier curves in computer graphics (Farin, 2001). For SIEs, the method involves approximating a "smoother" version of the unknown function after factoring out the singular weight function. This approach leverages the uniform convergence properties of Bernstein polynomials to approximate the non-singular part of the solution, offering an alternative pathway to handle the inherent difficulties of SIEs.

Yilmaz and Cetin (2017) compared the Bernstein piecewise polynomial method, Integral mean value method, Taylor series method and the Least-squares method to solve the integral equation of the second kind. Heidari *et al.* (2025) compared the Cauchy integral equation and the Legendre polynomial method and assessed them for convergence and error. Mosleh *et al.* (2011) introduces Bernstein polynomial expansion for weakly singular and Cauchy integral equation, and compared his result with the classical Legendre and Chebyshev method, showing Bernstein to have a superior numerical stability. In this study, the

Chebyshev, Legendre and Bernstein polynomial methods was utilized to solve singular integral equation of the second kind, unlike the work (Mosleh *et al.*, 2011) which solve for weakly singular integral equations. The three methods were compared for error and accuracy.

MATERIALS AND METHODS

This study utilizes MATLAB version 2015a for all solution formulations, result comparisons and plots.

Singular Integral Equations of the Second Kind

Singular integral equations (SIEs) of second type are characterized by the unknown function appearing both inside the integral sign and outside (Saira, 2019). The SIEs is given as

$$a u(x) + \frac{b}{\pi} \oint_{-1}^1 \frac{u(y)k(x,y)}{(y-x)} dy = f(x) \dots\dots\dots (1)$$

where a and b are constants. $k(x, y)$ is a known kernel, and $f(x)$ is a known function (Chakrabati, 2012)

Note: The coefficients and kernels in these equations are generally assumed to satisfy a Hölder Condition, a smoothness requirement that the unknown functions are also expected to fulfill (Chakrabati, 2012).

The major characteristic of singular integral equations of the second type is their capacity to preserve the essential features of the unknown function's singularity.

This implies that the solution $u(x)$ itself may exhibit singular behaviour at the integration interval's endpoints, often expressible as a product of a weight function $w(x)$ representing the boundary singularities and a smoother function $g(x)$.

That is

$$u(x) = w(x)g(x)$$

where: $w(x) = (x+1)^\alpha(1-x)^\beta$

The endpoints, whether bounded or unbounded, are governed by an 'index' (k), which is linked intrinsically to the exponent of the function. This index typically assumes values of -1 , 0 or 1 , dictating distinct solution characteristics: if $k = 1$, the solution $u(x)$ is unbounded at both endpoints $x = \pm 1$; if $k = 0$, it is bounded at one endpoint and unbounded at the other; and if $k = -1$, the solution $u(x)$ is bounded at both endpoints (Erdogan, 1969).

Algorithm

Consider the general second-kind singular integral equation (SIE) of Cauchy type:

$$\lambda u(x) + \frac{1}{\pi} \text{P.V.} \oint_{-1}^1 \frac{u(t)}{(t-x)} dt = f(x), \quad x \in (-1, 1) \dots\dots\dots (2)$$

Where: λ is a known constant, $u(x)$ is the unknown function, and $f(x)$ is the known forcing function

Algorithm 1: Chebyshev Polynomial Method (Weighted Collocation)

This algorithm leverages the analytical identity of the weighted Chebyshev polynomials ($T_n(x)$) to resolve the Cauchy singularity, which is the key to its spectral accuracy.

- 1. Select degree N :** Choose the degree of approximation N . The resulting linear system will be an $(N+1) \times (N+1)$ system.
- 2. Weighted approximation:** Approximate the unknown function $u(x)$ using the weighted Chebyshev series, where the weight factor naturally accommodates the solution's singular behaviour:

$$u_N(x) = \frac{1}{\sqrt{1-x^2}} \sum_{n=0}^N a_n T_n(x) \dots\dots\dots (3)$$

- 3. Apply analytical identity:** Substitute $u_N(x)$ into the integral equation and utilize the fundamental identity for the Cauchy integral of a weighted Chebyshev polynomial, which converts the singular integral term into a sum involving Chebyshev polynomials of the second kind ($U_n(x)$):

$$\frac{1}{\pi} \text{P.V.} \int_{-1}^1 \frac{T_n(t)}{\sqrt{(1-t^2)(t-x)}} dt = -U_{n-1}(x), \quad n \leq 1 \dots\dots\dots (4)$$

(Note: The term for $n = 0$ is zero, $U_{-1}(x) = 0$).

- 4. Form algebraic equation:** The SIE is transformed into the following algebraic equation:

$$\lambda \frac{1}{(\sqrt{1-x^2})} \sum_{n=0}^N a_n T_n(x) - \sum_{n=1}^N a_n U_{n-1}(x) = f(x) \dots\dots\dots (5)$$

- 5. Select collocation points:** Choose $N+1$ collocation points x_j , which are the zeros of $T_{N+1}(x)$ (Chebyshev roots):

$$x_j = \cos\left(\frac{(2j+1)\pi}{2(N+1)}\right), \quad j = 0, 1, \dots, N \dots\dots\dots (6)$$

- 6. Construct linear system:** Substitute the collocation points x_j into the algebraic equation to construct the linear system $Ma = f$.

i. Matrix Elements ($M_{j,n}$):

$$M_{j,n} = \lambda \frac{T_n(x_j)}{\sqrt{1-x_j^2}} - \begin{cases} u_{n-1}x_j & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$$

ii. Right-Hand Side (f_j): $f_j = f(x_j)$.

- 7. Solve:** Solve the linear system for the unknown coefficient vector $a = (a_0, a_1, \dots, a_N)^T$.

8. **Final solution:** The approximate solution is recovered from the series $u_N(x)$. (Christiansen *et al.*, 2023)

Algorithm 2: Legendre Polynomial Method (Unweighted Collocation)

This algorithm uses the standard Legendre basis ($P_n(x)$), which requires the singular integral to be calculated numerically using specialized quadrature at the collocation points.

1. **Select degree N:** Choose the degree of approximation N.
2. **Unweighted approximation:** Approximate the unknown function $u(x)$ using the standard Legendre series:

$$u_N(x) = \sum_{n=0}^N b_n P_n(x) \quad \dots \dots (7)$$

3. **Define singular integral term:** Define the singular integral term for the basis function $P_n(t)$:

$$S_n(x) = \frac{1}{\pi} \text{P.V.} \int_{-1}^1 \frac{P_n(t)}{t-x} dt$$

4. **Form discretized equation:** Substitute $u_N(x)$ into the SIE to get the equation in terms of b_n :

$$\sum_{n=0}^N b_n [\lambda P_n(x) + S_n(x)] \approx f(x)$$

5. **Select collocation points:** Choose $N + 1$ collocation points x_j (e.g., the roots of $P_{N+1}(x)$ or uniform points).
6. **Numerical Quadrature:** For all $j, n = 0, \dots, N$, numerically evaluate the singular integral $S_n(x_j)$ using a dedicated Cauchy Principal Value Quadrature formula.
7. **Construct Linear System:** Use the numerically calculated values to form the linear system $Mb = f$.

- i Matrix Elements ($M_{j,n}$):

$$M_{j,n} = \lambda P_n(x_j) + S_n(x_j)$$

- ii Right-Hand Side (f_j): $f_j = f(x_j)$

8. **Solve:** Solve the linear system for the unknown coefficient vector $b = (b_0, b_1, \dots, b_N)^T$.
9. **Final solution:** The approximate solution is recovered from the series $u_N(x)$. (Legendre, 1785)

Algorithm 3: Bernstein Polynomial Method (Transformation & Collocation)

This method uses the non-orthogonal Bernstein basis, requiring a domain transformation and handling the singularity purely through numerical methods, which typically results in a dense matrix.

1. **Select degree N:** Choose the degree of approximation N.
2. **Domain transformation:** Define the change of variable to map the integral domain $[-1, 1]$ to the standard Bernstein domain $[0, 1]$:

$$z = \frac{x+1}{2}$$

3. **Bernstein approximation:** Approximate $u(x)$ using the transformed Bernstein polynomials $B_{N,k}(z)$:

$$u_N(x) = \sum_{k=0}^N c_k B_{N,k}\left(\frac{x+1}{2}\right)$$

4. **Define singular integral term:** Define the singular integral term for the Bernstein basis function:

$$I_k(x) = \frac{1}{\pi} \text{P.V.} \int_{-1}^1 \frac{B_{N,k}\left(\frac{t+1}{2}\right)}{t-x} dt$$

5. **Select collocation points:** Choose $N + 1$ collocation points x_j in $(-1, 1)$ (often uniform or Chebyshev points).
6. **Numerical quadrature:** For all $j, k = 0, \dots, N$, numerically evaluate the singular integral $I_k(x_j)$ using an appropriate P.V. quadrature formula.
7. **Construct linear system:** Use the numerically calculated values to form the linear system $Rc = f$.

- i Matrix Elements ($R_{j,k}$):

$$R_{j,k} = \lambda B_{N,k}\left(\frac{x_j+1}{2}\right) + I_k(x_j)$$

- ii Right-Hand Side (f_j): $f_j = f(x_j)$

8. **Solve:** Solve the linear system for the unknown coefficient vector $c = (c_0, c_1, \dots, c_N)^T$.
9. **Final solution:** The approximate solution is recovered from the series $u_N(x)$. (Bhat *et al.*, 2024)

RESULTS AND DISCUSSION

Here, we apply each of the three methods: Legendre, Chebyshev and Bernstein polynomial methods to solve three examples and their results were compared for accuracy and error. These examples are chosen based on convenience and their analytic result is presented in the work of (Muskhelishvili & Radok, 2008). These methods were implemented using MatLab and the major steps are highlighted below.

Example 1 (Linear Kernel)

Consider the singular integral equation of second kind:

$$u(x) + \frac{1}{\pi} \int_{-1}^1 \frac{u(y)}{y-x} dy = f(x), \quad \text{for } x \in (-1, 1)$$

Here, let $f(x) = x$ and the exact solution $u(x) = u(x) = \sqrt{\frac{1}{1-x^2}}$

This represents the case where the solution is unbounded at its endpoints ($x = \pm 1$).

Step-by-Step Numerical Solution

These steps are stated according to our method in the algorithm and implemented using MatLab.

Approximate Solution and Comparison with Exact Solution

Table 1 presents the numerical results for *example 1* for $N = 10$ at various points in the interval $[-1, 1]$. The analytical solution was obtained from the work of

(Muskhelishvili & Radok, 2008) and the solutions for Legendre, Chebyshev and Bernstein polynomials were obtained using MatLab simulation.

It can be seen obviously that for $N = 10$ in the interval $[-1, 1]$, given as $-0.9, -0.7, -0.5, -0.3, -0.1, 0.1, 0.3, 0.5, 0.7, 0.9$, the Chebyshev polynomial method performs better. Followed by the Legendre polynomial method and then the Bernstein polynomial method. The absolute error $|u(x) - u_N(x)|$, is calculated for each method at three discrete parts to quantify the accuracy.

Table 1: Comparison of numerical results for Example 1: $u(x) = T_1(x)/\omega(x)$ ($N=10$)

x	Exact $u(x)$	Chebyshev $u_N(x)$	Legendre $u_N(x)$	Bernstein $u_N(x)$
-0.9	-2.064742×10^0	-2.064742×10^0	-1.461632×10^0	-1.461632×10^0
-0.7	-9.801961×10^{-1}	-9.801961×10^{-1}	-3.848415×10^{-1}	-3.848415×10^{-1}
-0.5	-5.773503×10^{-1}	-5.773503×10^{-1}	-6.623926×10^{-3}	-6.623926×10^{-3}
-0.3	-3.144855×10^{-1}	-3.144855×10^{-1}	1.246475×10^{-1}	1.246475×10^{-1}
-0.1	-1.005038×10^{-1}	-1.005038×10^{-1}	9.035034×10^{-2}	9.035034×10^{-2}
0.1	1.005038×10^{-1}	1.005038×10^{-1}	-1.770071×10^{-1}	-1.770071×10^{-1}
0.3	3.144855×10^{-1}	3.144855×10^{-1}	-8.701615×10^{-1}	-8.701615×10^{-1}
0.5	5.773503×10^{-1}	5.773503×10^{-1}	-1.570669×10^0	-1.570669×10^0
0.7	9.801961×10^{-1}	9.801961×10^{-1}	-9.706156×10^{-1}	-9.706156×10^{-1}
0.9	2.064742×10^0	2.064742×10^0	1.451962×10^0	1.451962×10^0

Table 1 and Table 2 show the computed values and comparison of the absolute errors associated with the three orthogonal polynomial methods and the exact solution respectively. This comparison in Table 2, show that the Chebyshev polynomial method with an average absolute error of 1.19×10^{-15} has higher accuracy than both Legendre polynomial (1.31) and Bernstein polynomial method (1.31). While, the Legendre polynomial and the Bernstein polynomial method have almost similar average error, the Chebyshev polynomial method has a far lesser average absolute error. Considering $N = 10$ on a closed interval $[-1, 1]$, further buttressed our conclusion that the Chebyshev polynomial method performs better than the two other methods mentioned above, agreeing with existing literature.

Table 2: Absolute error for Example 1: $u(x) = T_1(x)/\omega(x)$ ($N=10$)

x	Err Cheb	Err Leg	Err Bern
-0.9000	1.33×10^{-15}	6.03×10^{-01}	6.03×10^{-01}
-0.7000	7.77×10^{-16}	5.95×10^{-01}	5.95×10^{-01}
-0.5000	45.55×10^{-17}	5.71×10^{-01}	5.71×10^{-01}
-0.3000	1.44×10^{-15}	4.39×10^{-01}	4.39×10^{-01}
-0.1000	1.39×10^{-16}	1.91×10^{-01}	1.91×10^{-01}
0.1000	9.71×10^{-17}	2.78×10^{-01}	2.78×10^{-01}
0.3000	9.99×10^{-16}	$1.18 \times 10^{+00}$	$1.18 \times 10^{+00}$
0.5000	1.22×10^{-15}	$2.15 \times 10^{+00}$	$2.15 \times 10^{+00}$
0.7000	1.11×10^{-15}	$1.95 \times 10^{+00}$	$1.95 \times 10^{+00}$
0.9000	1.33×10^{-15}	6.13×10^{-01}	6.13×10^{-01}
Average Absolute Error	1.19×10^{-15}	1.31×10^{00}	1.31×10^{00}

Figure 1 shows a graphical comparison of the solution for Example 1, using Legendre, Chebyshev and Bernstein polynomial methods with the exact solution for $N = 10$ on the interval $[-1, 1]$. It clearly shows how the Chebyshev polynomial outperformed both the Legendre polynomial and the Bernstein polynomial method in comparison with the exact solution.

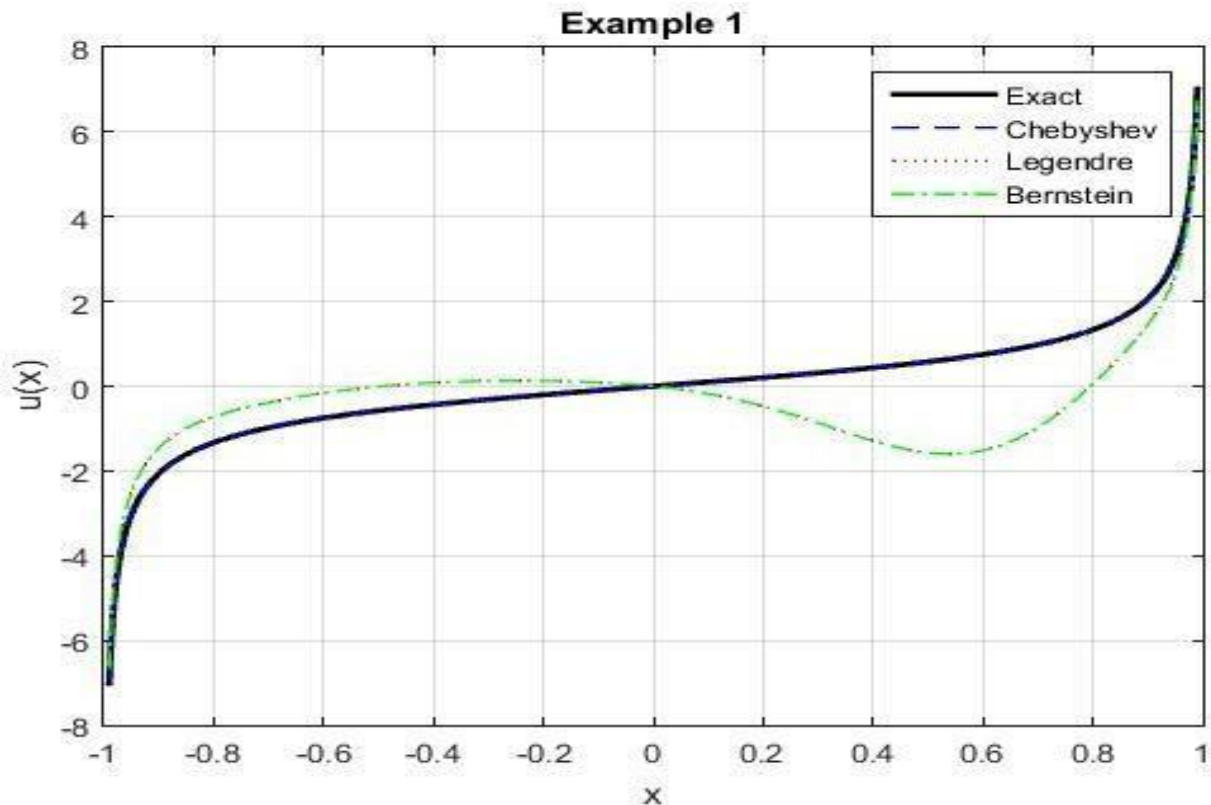


Figure 1: Comparison of Example 1 using the three different methods (N=5)

Table 3: Comparison of numerical results for Example 2: $u(x) = T_0(x)/\omega(x)$ (N=10)

x	Exact $u(x)$	Chebyshev $u_N(x)$	Legendre $u_N(x)$	Bernstein $u_N(x)$
-0.9	2.294157×10^0	2.294157×10^0	2.026076×10^0	2.026076×10^0
-0.7	1.400280×10^0	1.400280×10^0	1.033120×10^0	1.033120×10^0
-0.5	1.154701×10^0	1.154701×10^0	6.863030×10^{-1}	6.863030×10^{-1}
-0.3	1.048285×10^0	1.048285×10^0	4.284000×10^{-1}	4.284000×10^{-1}
-0.1	1.005038×10^0	1.005038×10^0	1.627254×10^{-1}	1.627254×10^{-1}
0.1	1.005038×10^0	1.005038×10^0	-2.013020×10^{-1}	-2.013020×10^{-1}
0.3	1.048285×10^0	1.048285×10^0	-6.985091×10^{-1}	-6.985091×10^{-1}
0.5	1.154701×10^0	1.154701×10^0	-9.134487×10^{-1}	-9.134487×10^{-1}
0.7	1.400280×10^0	1.400280×10^0	-6.654309×10^{-2}	-6.654309×10^{-2}
0.9	2.294157×10^0	2.294157×10^0	1.803189×10^0	1.803189×10^0

Example 2 (Identity Kernel)

We consider the homogeneous version of the standard Cauchy Singular Integral Equation (SIE) with $\lambda = 1$ and $f(x) = 0$, which has known non-trivial solutions:

$$u(x) + \frac{1}{\pi} \text{P.V.} \int_{-1}^1 \frac{u(t)}{t-x} dt = 0 \quad \dots \dots (8)$$

The general solution $u(x)$ to this full singular equation is known to be proportional to the weighted function $w(x)$

$$= w(x) = \sqrt{\frac{1}{1-x^2}}$$

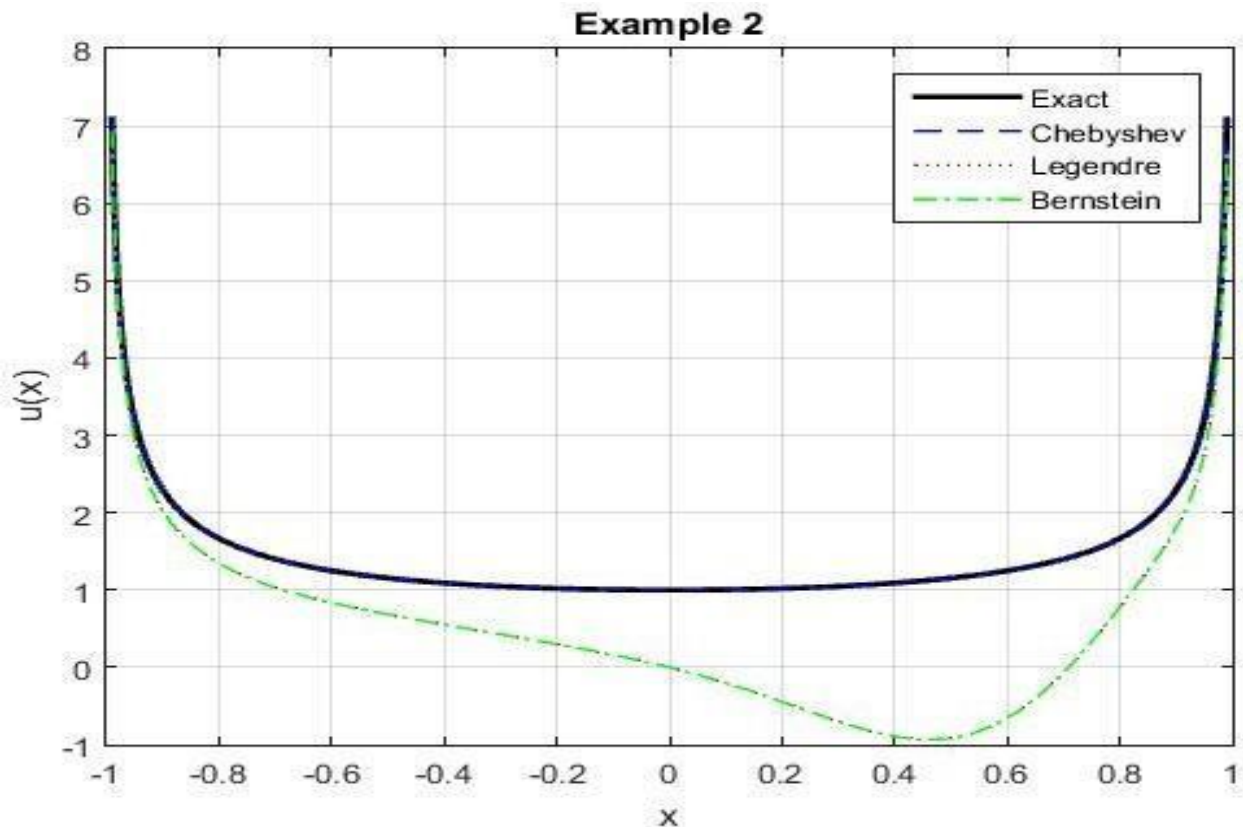
polynomial methods and the exact solution, respectively. This comparison shows that the Chebyshev polynomial method with an average absolute error of 5.68×10^{-18} has higher accuracy than both Legendre polynomial and Bernstein polynomial method (8.94×10^{-01}). While the Legendre polynomial and the Bernstein polynomial methods are powerful numerical methods for solving SIEs-2, the Chebyshev polynomial method gives machine-level accuracy considering $N = 10$ on a closed interval $[-1, 1]$. It went further to buttress our conclusion that the Chebyshev polynomial method performs better than the two other methods mentioned above, also agreeing with existing literature.

Tables 3 and 4 show the computed values and the absolute error comparison associated with the three

Table 4: Absolute Error for Example 2: $u(x) = T_2(x)/\omega(x)$ (N=10)

x	Cheb Err	Leg Err	Bern Err
-0.9000	1.33×10^{-15}	2.68×10^{-01}	2.68×10^{-01}
-0.7000	1.33×10^{-15}	3.67×10^{-01}	3.67×10^{-01}
-0.5000	0.00×10^{00}	4.68×10^{-01}	4.68×10^{-01}
-0.3000	4.44×10^{-16}	6.20×10^{-01}	6.20×10^{-01}
-0.1000	6.66×10^{-16}	8.42×10^{-01}	8.42×10^{-01}
0.1000	2.22×10^{-16}	$1.21 \times 10^{+00}$	1.21×10^{-01}
0.3000	0.00×10^{00}	$1.75 \times 10^{+00}$	$1.75 \times 10^{+00}$
0.5000	0.00×10^{00}	$2.07 \times 10^{+00}$	$2.07 \times 10^{+00}$
0.7000	4.44×10^{-16}	$1.47 \times 10^{+00}$	$1.47 \times 10^{+00}$
0.9000	1.33×10^{-15}	4.91×10^{-01}	4.91×10^{-01}
Average Abs. Err.	5.68×10^{-16}	8.94×10^{-01}	8.94×10^{-01}

Figure 2 shows a graphical comparison of the solution of Example 1, using Legendre, Chebyshev and Bernstein polynomial methods with the exact solution for $N = 10$ on the interval $[-1, 1]$. It clearly shows how the Chebyshev polynomial outperformed both the Legendre polynomial and the Bernstein polynomial method in comparison with the exact solution.

**Figure 2: Comparison of Example 2 using the three different methods (N=10)****Table 5: Comparison of numerical results for Example 3: $u(x) = T_2(x)/\omega(x)$ (N=10)**

x	Exact $u(x)$	Chebyshev $u_N(x)$	Legendre $u_N(x)$	Bernstein $u_N(x)$
-0.9	1.422378×10^0	1.422378×10^0	9.218972×10^{-1}	9.218972×10^{-1}
-0.7	-2.800560×10^{-2}	-2.800560×10^{-2}	-2.594209×10^{-1}	-2.594209×10^{-1}
-0.5	-5.773503×10^{-1}	-5.773503×10^{-1}	-4.156834×10^{-1}	-4.156834×10^{-1}
-0.3	-8.595936×10^{-1}	-8.595936×10^{-1}	-2.933235×10^{-1}	-2.933235×10^{-1}
-0.1	-9.849371×10^{-1}	-9.849371×10^{-1}	-1.042825×10^{-1}	-1.042825×10^{-1}
0.1	-9.849371×10^{-1}	-9.849371×10^{-1}	5.024284×10^{-2}	5.024284×10^{-2}
0.3	-8.595936×10^{-1}	-8.595936×10^{-1}	-3.426409×10^{-1}	-3.426409×10^{-1}
0.5	-5.773503×10^{-1}	-5.773503×10^{-1}	-1.464496×10^0	-1.464496×10^0
0.7	-2.800560×10^{-2}	-2.800560×10^{-2}	-1.856662×10^0	-1.856662×10^0
0.9	1.422378×10^0	1.422378×10^0	7.468242×10^{-1}	7.468242×10^{-1}

Example 3: Hilbert Kernel Variation (Symmetric Kernel)

We examine a non-homogeneous variation that involves a symmetric singular kernel $\frac{1}{t+x}$, which tests the methods' ability to exploit symmetry.

$$u(x) + \int_{-1}^1 \frac{u(t)}{t+x} dt = \frac{\pi x}{2} \quad \dots \dots (9)$$

The exact solution for this specific right-hand side is $u(x) = \frac{\pi}{2} \sqrt{1-x^2}$.

Table 6 shows that the Chebyshev, Legendre and Bernstein polynomial methods have an average absolute error of 9.45×10^{-16} , 1.16, 1.16, respectively. This highlights the superiority of the Chebyshev polynomial method over the other methods, making it the most suitable method for solution for SIEs-2 with Symmetric Kernel.

Table 6: Absolute error for Example 3: $u(x) = T_0(x)/\omega(x)$ (N=10)

x	Cheb Err	Leg Err	Bern Err
-0.9000	0.00×10^0	5.00×10^{-01}	5.00×10^{-01}
-0.7000	1.35×10^{-15}	2.31×10^{-01}	2.31×10^{-01}
-0.5000	6.66×10^{-16}	1.62×10^{-01}	1.62×10^{-01}
-0.3000	3.33×10^{-16}	5.66×10^{-01}	5.66×10^{-01}
-0.1000	0.00×10^0	8.81×10^{-01}	8.81×10^{-01}
0.1000	2.22×10^{-16}	1.04×10^0	1.04×10^0
0.3000	2.22×10^{-16}	5.17×10^{-01}	5.17×10^{-01}
0.5000	0.00×10^0	8.87×10^{-01}	8.87×10^{-01}
0.7000	1.44×10^{-15}	1.83×10^0	1.83×10^0
0.9000	6.66×10^{-16}	6.76×10^{-01}	6.76×10^{-01}
Average Abs. Err.	9.45×10^{-16}	1.16×10^0	1.16×10^0

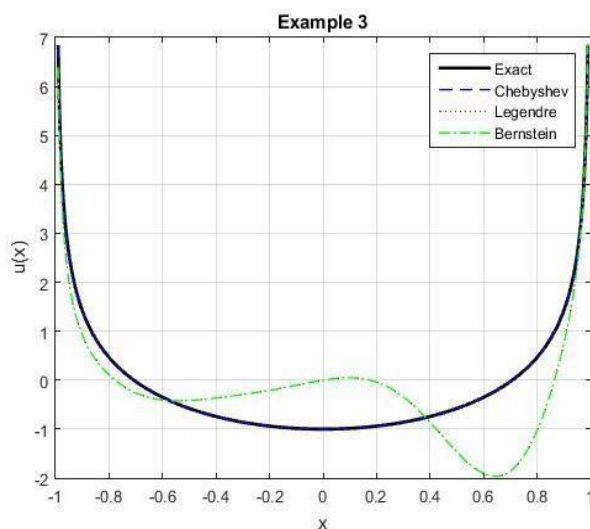


Figure 3: Comparison of Example 3 using the three different methods (N=10)

Figure 3 shows the graph of the three polynomial methods against the exact solution. This shows an almost perfect conformity between the Chebyshev polynomial result and the exact value.

CONCLUSION

A comparative analysis of three numerical methods for solving singular integral equations (SIEs) of the second kind: the Legendre, Chebyshev, and Bernstein polynomial methods was studied. SIEs are characterized by an unbounded kernel function which present significant challenges in mathematical modeling across various scientific and engineering disciplines. A common approach to solving them numerically is to approximate the unknown function with a series of orthogonal polynomials, which transforms the problem into a system of linear algebraic equations. In the MatLab coding, Both the Legendre and Bernstein methods rely on the same numerical method which is a discrete collocation approach using a highly accurate Chebyshev-Gauss-Lobatto (CGL) quadrature rule to approximate the singular integral term $S[u](x)$.

In contrast, the Chebyshev polynomial method uses the Weighted Basis and then analysis formula to solve its singularity problem thereby eliminating the quadrature error. By applying each of these three methods to a benchmark SIEs with a known exact solution, we evaluated and compared their performance in terms of accuracy and convergence. The implementation was carried out using MatLab, and the results which show that Chebyshev polynomial method is superior to the other methods, were meticulously analyzed to quantify the absolute error of each method.

Based on the numerical results and error analysis, the study concludes that the Chebyshev polynomial method is the most effective and accurate approach for solving the class of SIEs examined. Its superior performance is directly linked to the properties of Chebyshev polynomials, which are orthogonal with respect to a non-uniform weight function that naturally aligns with the typical singular behavior of solutions at the integration endpoints. The absolute error comparison table demonstrated that the Chebyshev method consistently yielded the lowest error, particularly in regions close to the singularities where the other methods showed less accuracy. The Legendre and Bernstein methods, while viable, proved to be less robust in handling the endpoint singularities, leading to larger discrepancies between their approximate solutions and the exact solution and this conform to the findings of (Mohamed and Taher, 2017), which opined that the Chebyshev polynomial method is better than the Legendre polynomial method for solving SIEs-2.

These three methods for solving singular integral equations as discuss in this study and the examples used highlights Chebyshev polynomial method to be a superior method than Bernstein and Legendre polynomial methods. It is not a generalization that the Chebyshev method will outperform the other methods using different examples. Further studies can be carried out using different examples to verify the superiority of the Chebyshev method. Other methods like spectral and projection method, Galerkin and Tau method can be utilized to improve the Bernstein or the Legendre method and compared with the Chebyshev method.

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