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A STUDY ON CAYLEY DIGRAPHS OF THE FULL TRANSFORMATION SEMIGROUP WITH RESPECT TO GREEN'S D-CLASSES

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ABSTRACT
Let T_n be the full transformation semigroup on an n-element set and let D_r be the Green D-class of all transformations of rank *r*. We investigate the looped simple Cayley digraph $\Gamma_{n,r} = Cay(T_n, D_r)$ under right multiplication. The key difficulty is that the standard kernel-refinement criterion only guarantees a factorization $\beta = \alpha\gamma$ with an unrestricted factor γ; it does not determine when the factor can be chosen with prescribed rank *r*. We solve this extension problem exactly and prove that $\alpha \to \beta$ if and only if $\ker(\alpha)$ refines $\ker(\beta)$ and $\text{rank}(\beta) \le r \le \text{rank}(\beta) + n - \text{rank}(\alpha)$. From this criterion we obtain closed out-degree formulae and a new in-degree formula governed by the block-size profile of $\ker(\beta)$. We identify a source-core decomposition in which all vertices of rank greater than *r* have in-degree zero and every arc enters the ideal of transformations of rank at most *r*. We classify all strongly connected components, determine the condensation digraph and its exact height, and derive the exact directed distance from an arbitrary transformation to the constant maps. Weak connectivity, local cyclicity and isomorphism rigidity are then obtained as consequences.

Keywords: Full transformation semigroup; Cayley digraph; Green's relations; Rank; Connectivity

INTRODUCTION
Cayley graphs encode algebraic multiplication as graph adjacency and have long provided a bridge between algebra and combinatorics, beginning with Cayley's graphical representation of finite groups (Cayley, 1978). For semigroups, directed Cayley graphs are especially natural because multiplication need not be invertible. Their reachability, component, diameter and symmetry properties may therefore reflect ideal structure, Green's relations and the failure of cancellation. General investigations of semigroup Cayley graphs and their connectedness include: Kelarev (2006), Luo *et al.* (2011), Suksumran and Panma (2015), Tisklang and Panma (2018), while Green-type symmetries and graph-semigroup interactions have been examined from related viewpoints in Parks (2011), Yang and Yang (2009).

The full transformation semigroup T_n is the semigroup of all self-maps of an n-element set. It is a fundamental finite semigroup whose structure is governed by image sets, kernel partitions and ranks; standard accounts include: Clifford and Preston (1961), Ganyshkin and Mazorchuk (2009), Grillet (1995), Higgins (1978), Howie (1995). Green's foundational relations Green (1951) organizes T_n into classes determined by image, kernel and rank. In particular, the D-classes coincide with the rank classes. Classical work on generators, idempotent generation and rank in finite transformation semigroups further shows how strongly the rank filtration controls the algebraic structure (Gomes & Howie, 1987; Howie, 1978; Howie & McFadden, 1990). Transformation semigroups also arise naturally in automata theory and finite dynamical systems (Eilenberg, 1974; Harary, 1959).

More recently, digraphs have been used both to generate transformation semigroups and to analyze their algorithmic and structural behaviours (Cameron *et al.*, 2016; East *et al.*, 2017; Eze *et al.*, 2025). Computational enumeration has become an important complementary tool in this area, especially for small degrees (East *et al.*, 2014). Other graph constructions attached directly to the full transformation semigroup are considered in Jeff *et al.* (2022), while centralizers and related internal symmetries of T_n are studied in Araujo and Konieczny (2013).

A particularly relevant recent contribution is the study of Cayley digraphs of T_n with connection sets derived from Green's L- and R-classes (Nupo, & Chaiya, 2023). This raises the natural question of what happens when the more symmetric rank class D_r is used as the connection set. The resulting digraph contains all n^n transformations as vertices and records right multiplication by every transformation of rank r.

A subtle point arises in this setting. The familiar factorization condition $\beta \in \alpha T_n$ determines whether $\beta = \alpha\gamma$ for some transformation γ, but it does not determine whether such a factor can be chosen to have a prescribed rank r. The latter is an extension problem for a partially defined transformation on $\text{Im}(\alpha)$. Ignoring this extension constraint leads to incorrect adjacency statements, particularly when α is a permutation, because then the right factor is uniquely determined.

This paper resolves that issue and develops a structural and enumerative theory that remains within the original Green-D-class setting. To the best of our knowledge, the prescribed-rank factorization problem, the two-sided degree profiles and the condensation structure have not previously been determined for this family. The principal

contributions are: an exact necessary-and-sufficient adjacency criterion; a closed out-degree formula; an in-degree formula that detects the block-size profile of the terminal kernel; a source-core decomposition; a complete classification of strongly connected components and the condensation digraph; an exact distance formula to the constant transformations; and connectivity and isomorphism consequences. Exhaustive computation for $n \leq 4$ provides an independent check. Thus, the work extends the Green-class programme of Nupo and Chaiya (2023) from L- and R-classes to the more symmetric D-classes and adds graph invariants not visible from rank alone.

Preliminaries

Throughout, let $n \geq 2$ and let $X_n = \{1, \ldots, n\}$. The full transformation semigroup T_n consists of all maps $\alpha : X_n \to X_n$. Transformations act on the right, so

$$x(\alpha\delta) = (x\alpha)\delta \quad (x \in X_n).$$

For $\alpha \in T_n$, write $\mathrm{Im}(\alpha) = \{x\alpha : x \in X_n\}$ and $\mathrm{rank}(\alpha) = |\mathrm{Im}(\alpha)|$. Its kernel partition is $\ker(\alpha) = \{\alpha^{-1}(y) : y \in \mathrm{Im}(\alpha)\}$. For partitions π and ρ of X_n, we write $\pi \preceq \rho$ when π refines ρ.

The Green relations in T_n satisfy

$$\alpha \, L \, \beta \iff Im(\alpha) = Im(\beta), \quad \alpha \, R \, \beta \iff ker(\alpha) = ker(\beta),$$

and

$$\alpha \, D \, \beta \iff rank(\alpha) = rank(\beta).$$

Accordingly, $D_r = \{\delta \in T_n : \mathrm{rank}(\delta) = r\}$, for $1 \leq r \leq n$. The number of rank-a transformations is

$$|D_a| = C(n,a) \cdot a! \cdot S(n,a) = (n)_a \cdot S(n,a), \quad (1)$$

where $S(n,a)$ is a Stirling number of the second kind and $(n)_a = n\,(n-1)\cdots(n-a+1)$.

Definition 2.1: For $1 \leq r \leq n$, the rank-class Cayley digraph $\Gamma_{n,r} = \mathrm{Cay}(T_n, D_r)$ has vertex set T_n and an arc $\alpha \to \beta$ if and only if $\beta = \alpha\delta$ for some $\delta \in D_r$. Parallel arcs are suppressed and loops are retained.

A digraph is weakly connected if its underlying undirected graph is connected. It is locally cyclic if every arc belongs to a directed cycle. This property is sometimes called local connectedness in the semigroup-Cayley-digraph literature; we use "locally cyclic" to avoid confusion with the standard graph-theoretic meaning of local connectivity.

The following standard factorization criterion is a basic consequence of the description of principal right ideals in T_n; see, for example, [8, 15, 20].

Lemma 2.2: For $\alpha, \beta \in T_n$, there exists $\gamma \in T_n$ such that $\beta = \alpha\gamma$ if and only if $\ker(\alpha) \preceq \ker(\beta)$.

Proof: If $\beta = \alpha\gamma$ and $x\alpha = y\alpha$, then $x\beta = (x\alpha)\gamma = (y\alpha)\gamma = y\beta$, so $\ker(\alpha) \preceq \ker(\beta)$. Conversely, assume the refinement. Define $\theta : \mathrm{Im}(\alpha) \to X_n$ by $(x\alpha)\theta = x\beta$. The refinement makes θ well defined. Extending θ arbitrarily to X_n gives a transformation γ satisfying $\beta = \alpha\gamma$.

Lemma 2.3: Let $A \subseteq X_n$ with $|A| = a$, and let $\theta : A \to X_n$ have image size b. Then θ extends to a transformation $\delta \in T_n$ of rank r if and only if $b \leq r \leq b + n - a$. (2)

Proof: Any extension has image containing $\mathrm{Im}(\theta)$, so its rank is at least b. The $n - a$ points outside A can introduce at most $n - a$ additional image values, so the rank is at most $b + n - a$. Conversely, suppose (2) holds. Choose $r - b$ distinct values in $X_n \setminus \mathrm{Im}(\theta)$ and assign them to $r - b$ distinct points of $X_n \setminus A$. Map every remaining point of $X_n \setminus A$ to any value already used. The resulting extension has rank exactly r.

Remark 2.4: For every arc $\alpha \to \beta$ in $\Gamma_{n,r}$,

$$rank(\beta) = rank(\alpha\delta) \leq min\{rank(\alpha), r\}.$$

Thus, rank is non-increasing along directed paths.

RESULT AND DISCUSSION

Exact Adjacency and Degree Profiles

Theorem 3.1.1: Let $\alpha, \beta \in T_n$, and put $a = \mathrm{rank}(\alpha)$ and $b = \mathrm{rank}(\beta)$. Then $\alpha \to \beta$ in $\Gamma_{n,r}$ if and only if

$$ker(\alpha) \preceq ker(\beta) \text{ and } b \leq r \leq b + n - a. \quad (3)$$

Proof: Assume $\beta = \alpha\delta$ with $\mathrm{rank}(\delta) = r$. Lemma 2.2 gives $\ker(\alpha) \preceq \ker(\beta)$. On $A = \mathrm{Im}(\alpha)$, the restriction $\delta|A$ has image $A\delta = \mathrm{Im}(\alpha\delta) = \mathrm{Im}(\beta)$, so it has rank b. Lemma 2.3 therefore gives $b \leq r \leq b + n - a$.

Conversely, assume (3): By Lemma 2.2, the map $\theta : \mathrm{Im}(\alpha) \to X_n$, $(x\alpha)\theta = x\beta$, is well defined and has image $\mathrm{Im}(\beta)$, hence rank b. The rank inequalities in (3) and Lemma 2.3 provide an extension $\delta \in T_n$ of rank r. Then $\beta = \alpha\delta$, so $\alpha \to \beta$.

Corollary 3.1.2: If $\mathrm{rank}(\alpha) = a$, then the possible ranks of out-neighbours of α are exactly

$$max\{1, a + r - n\} \leq b \leq min\{a, r\}. \quad (4)$$

Proof: The upper bound follows from Remark 2.4; the lower bound is equivalent to $r \leq b + n - a$. Every integer b in the interval is realised by merging the blocks of $\ker(\alpha)$ into b blocks and assigning distinct image values to the resulting blocks.

Theorem 3.1.3: If $\mathrm{rank}(\alpha) = a$, then the out-degree of α in the looped simple digraph $\Gamma_{n,r}$ is

$$d^{+}_{n,r}(\alpha) = \Sigma\,(n)_b \cdot C(a,b), \text{ for } b = max\{1, a+r-n\} \text{ to } min\{a, r\}. \quad (5)$$

In particular, the out-degree depends only on $\mathrm{rank}(\alpha)$.

Proof: Every β satisfying $\ker(\alpha) \preceq \ker(\beta)$ is uniquely determined by the induced map $\theta : \mathrm{Im}(\alpha) \to X_n$ from Lemma 2.2. If $\mathrm{rank}(\beta) = b$, then θ is a rank-b map from an a-element set to an n-element set. There are $(n)_b \cdot C(a,b)$ such maps. Summing over the admissible ranks from Corollary 3.1.2 proves the formula.

Corollary 3.1.4: The number of arcs, including loops, is

$$|E(\Gamma_{n,r})| = \Sigma_{a=1}^{n} (n)_a \cdot S(n,a) \cdot d^{+}_{n,r}(a). \quad (6)$$

Proposition 3.1.5: A vertex α has a loop in $\Gamma_{n,r}$ if and only if $\mathrm{rank}(\alpha) \leq r$. Consequently, the number of looped vertices is

$$L_{n,r} = \Sigma_{a=1}^{r} (n)_a \cdot S(n,a). \quad (7)$$

Proof: Apply Theorem 3.1.1 with $\beta = \alpha$. The kernel condition is automatic, while the rank condition becomes $a \leq r \leq n$.

For $m \geq 1$, define the Bell polynomial $B_m(x) = \Sigma_{k=1}^{m} S(m,k) \cdot x^k$. If a rank-$b$ transformation β has kernel blocks of sizes $m_1, \ldots, m_b$, put

$$R_a(m_1, \ldots, m_b) = [x^a] \, \Pi_{(i=1)}\text{^}b \, B_{mi}(x). \quad (8)$$

Thus $R_a(m_1, \ldots, m_b)$ is the number of partitions of X_n into a blocks that refine $\ker(\beta)$.

Theorem 3.1.6: Let $\beta \in T_n$ have rank b and kernel-block sizes $m_1, \ldots, m_b$. If $b > r$, then $d^{-}_{n,r}(\beta) = 0$. If $b \leq r$, then

$$d^{-}_{n,r}(\beta) = \Sigma_{a=_}b\text{^}\{min\{n, n+b-r\}\} \, (n)_a \cdot R_a(m_1, \ldots, m_b). \quad (9)$$

Consequently, in-degree is generally determined by the kernel block-size profile and not by rank alone.

Proof: If $b > r$, no product by a rank-r transformation can have rank b, so the in-degree is zero. Assume $b \leq r$. An in-neighbor α of rank a must have a kernel partition refining $\ker(\beta)$, and Theorem 3.1.1 requires $b \leq r \leq b + n - a$, which is equivalent to $b \leq a \leq n + b - r$. For each admissible a, equation (8) counts the possible kernel partitions of α. Once such a partition with a block is fixed, there are $(n)_a$ injective assignments of image values to its blocks, and hence $(n)_a$ transformations with that kernel. Summing over a give (9).

Example 3.1.7: In $\Gamma_{4,2}$, a rank-2 transformation whose kernel has block sizes (3, 1) has in-degree

$$(4)_2 + 3(4)_3 + (4)_4 = 108,$$

whereas a rank-2 transformation of kernel type (2, 2) has in-degree

$$(4)_2 + 2(4)_3 + (4)_4 = 84.$$

Thus, the in-degree sequence distinguishes kernel shapes inside the same Green D-class.

Core, components and condensation

Let $I_r = \{\alpha \in T_n : \text{rank}(\alpha) \leq r\} = D_1 \cup \cdots \cup D_r$. This is the principal rank ideal of T_n.

Proposition 3.2.1: Every arc of $\Gamma_{n,r}$ has its terminal vertex in I_r. A vertex has in-degree zero if and only if its rank is greater than r. Hence every vertex outside I_r is a source, every outgoing arc from such a vertex enters I_r in one step, and every directed cycle is contained in the induced sub-digraph on I_r.

Proof: If $\beta = \alpha\delta$ with $\text{rank}(\delta) = r$, then $\text{rank}(\beta) \leq r$. Thus, no vertex outside I_r has an in-neighbor. Conversely, every vertex in I_r has a loop by Proposition 3.1.5, and therefore has positive in-degree. The remaining assertions follow immediately.

Theorem 3.2.2: Let $\alpha \in T_n$ have rank a.

(i) If $a \leq r$, then the strongly connected component containing α is its Green R-class $R\alpha = \{\beta \in T_n : \ker(\beta) = \ker(\alpha)\}$. Moreover, the sub-digraph induced by $R\alpha$ is complete and looped.

(ii) If $a > r$, then $\{\alpha\}$ is a singleton strongly connected component and α has no loop.

Proof: Suppose first that $a \leq r$. If $\beta \in R\alpha$, then $\text{rank}(\beta) = a$, the kernel condition in Theorem 3.1.1 holds in both directions, and $a \leq r \leq a + n - a = n$. Hence $\alpha \longrightarrow \beta$ and $\beta \longrightarrow \alpha$.

Conversely, if α and β are mutually reachable, rank monotonicity forces $\text{rank}(\alpha) = \text{rank}(\beta)$. Along a path, kernel partitions can only become coarser by Lemma 2.2. Equal ranks mean equal numbers of kernel blocks, so mutual reachability forces the kernels to coincide.

If $a > r$, every out-neighbor has rank at most $r < a$. No directed path can return to rank a, so the strongly connected component is the singleton $\{\alpha\}$. Proposition 3.1.5 shows that it has no loop.

Corollary 3.2.3: The number of strongly connected components is

$$scc(\Gamma_{n,r}) = \Sigma_{a=1}^{r} S(n,a) + \Sigma_{a=r+1}^{n} (n)_a \cdot S(n,a). \qquad (10)$$

Proof: For $a \leq r$, the rank-a R-classes are indexed by the $S(n,a)$ partitions of X_n into a block. For $a > r$, every rank-a transformation is a singleton component.

Theorem 3.2.4: Let $Q_{n,r}$ be the condensation digraph obtained by contracting every strongly connected component of $\Gamma_{n,r}$. Every arc between distinct vertices of $Q_{n,r}$ strictly decreases transformation rank. If $r < n$, the maximum length of a directed path in $Q_{n,r}$ is exactly r. If $r = n$, the condensation has no arcs between distinct components.

Proof: By Theorem 3.2.2, equal-rank mutually reachable vertices lie in the same R-class. If an arc joins distinct components and its initial rank is at most r, kernel refinement together with equality of ranks would force equal kernels and hence the same component. Therefore, the rank decreases. If the initial rank is greater than r, Proposition 4.1 makes the decrease immediate.

Assume $r < n$. A path beginning above rank r can use at most one edge before entering I_r, after which every inter-component edge reduces rank by at least one. Hence its length is at most $1 + (r - 1) = r$. This bound is attained: start with a permutation, move to a rank-r transformation, and then choose a chain of strictly coarsening kernel partitions having r, $r - 1$, …, 1 block. Theorem 3.1 realizes every step. If $r = n$, multiplication is by permutations and preserves the kernel, so every arc remains inside one R-class.

Theorem 3.2.5: Assume $r < n$, put $d = n - r$, and let α have rank a. The directed distance from α to the set D_1 of constant transformations is

$$dist(\alpha, D_1) = \lceil (a - 1)/(n - r) \rceil. \qquad (11)$$

For a prescribed constant χ_c, the same value is the distance when $a > 1$; when $a = 1$, the distance is 0 if $\alpha = \chi_c$ and 1 otherwise.

Proof: For an arc from rank a_i to rank a_{i+1}, Corollary 3.1.2 gives $a_{i+1} \geq a_i - d$. Thus, a path from rank a to rank 1 need at least $\lceil (a - 1)/d \rceil$ edges. Conversely, at each step choose an out-neighbor of rank $\max\{1, a_i - d\}$ by merging kernel blocks. The adjacency criterion permits this choice, and on the final step the unique image value can be prescribed as c. This attains the lower bound for $a > 1$. Constants form a complete looped R-class, giving the final assertion when $a = 1$.

Corollary 3.2.6: If $r < n$ and α has rank $a > 1$, then α has an out-neighbor of rank $\max\{1, a - (n - r)\}$. Repeating this construction gives a shortest path from α to D_1.

Proof: This is the constructive part of Theorem 3.2.5.

Theorem 3.2.7: For $n \geq 2$, $\Gamma_{n,r}$ is weakly connected if and only if $r < n$.

Proof: If $r < n$, Theorem 3.2.5 shows that every vertex has a directed path to a fixed constant transformation, so the underlying undirected graph is connected.

If $r = n$, then $D_n = S_n$. Right multiplication by a permutation preserves the kernel partition. Conversely, if two transformations have the same kernel, one is obtained from the other by right multiplication by a permutation. Hence the weak components are exactly the R-classes. Since $n \geq 2$, there is more than one kernel partition, so the digraph is not weakly connected.

Corollary 3.2.8: If $r = n$, the number of weak components is the Bell number $B_n = \Sigma_{a=1}^{n} S(n,a)$. If $r < n$, the weak diameter satisfies

$$diam_w(\Gamma_{n,r}) \leq 2\lceil (n - 1)/(n - r) \rceil. \qquad (12)$$

Proof: The first assertion follows from the description of the components in the proof of Theorem 3.2.7. For the second, Theorem 3.2.5 sends each of two vertices to the same prescribed constant in at most $\lceil (n-1)/(n-r) \rceil$ steps; reverse the second path in the underlying undirected graph.

Theorem 3.2.9: Every arc of $\Gamma_{n,r}$ belongs to a directed cycle if and only if r = n.

Proof: If r = n, every generator is a permutation. Whenever $\beta = \alpha\delta$, we have $\alpha = \beta\delta^{-1}$, so every arc lies on a directed 2-cycle (or is a loop).

If r < n, choose $\delta \in D_r$. There is an arc id $X_n \to \delta$, but no directed path from δ back to the identity because rank cannot increase from r to *n*.

Corollary 3.2.10: For $n \geq 2$, no $\Gamma_{n,r}$ is strongly connected or unilaterally connected.

Proof: If r < n, two distinct permutations cannot reach one another because every nontrivial outgoing step from a permutation has rank at most $r < n$. If r = n, distinct R-classes are mutually unreachable.

Theorem 3.2.11: For fixed $n \geq 2$, $\Gamma_{n,r} \cong \Gamma_{n,s} \iff r = s$.

Proof: The reverse implication is immediate. For the forward implication, a digraph isomorphism preserves the number of looped vertices. By Proposition 3.1.5, $L_{n,r} = \sum_{a=1}^{r} |D_a|$. Because every D_a is nonempty, $L_{n,r}$ is strictly increasing in r. Hence $L_{n,r} = L_{n,s}$ implies r = s.

Computational Verification and the Case $\Gamma_{3,2}$

The formulas were checked independently by exhaustive enumeration, following the now standard use of computation in the classification and enumeration of small transformation semigroups East *et al*. (2014). For each $n \leq 4$, all n^n transformations were generated. For every r, the program computed all products $\alpha\delta$ with $\delta \in D_r$, suppressed duplicate neighbors, and compared the resulting adjacency matrix with Theorem 3.1.1. It also verified the out-degree, in-degree, loop and component formulae, the source-core decomposition, and the exact distances to D_1. No discrepancy occurred. Table 1 records the principal invariants.

Table 1: Exhaustive verification for $2 \leq n \leq 4$. Arc counts include loops, and h(Q) is the maximum directed path length in the condensation

n	r	$\lvert T_n \rvert$	$\lvert D_r \rvert$	arcs	looped vertices	SCCs	h(Q)
2	1	4	2	8	2	3	1
2	2	4	2	8	4	2	0
3	1	27	3	81	3	25	1
3	2	27	18	279	21	10	2
3	3	27	6	153	27	5	0
4	1	256	4	1024	4	253	1
4	2	256	84	9136	88	176	2
4	3	256	144	13456	232	38	3
4	4	256	24	5056	256	15	0

For $\Gamma_{3,2}$, the three rank layers have sizes $|D_1| = 3$, $|D_2| = 18$ and $|D_3| = 6$. The in-degree formula adds information beyond this rank layering: vertices are grouped by the sizes of their kernel blocks, while the out-degree remains constant on each rank layer. Figure 1 retains a labelled diagram of representative vertices and arcs. It must be interpreted as a selected illustrative sub-digraph rather than the complete Cayley digraph: it contains all six permutations and all three constant maps, but only six of the 18 rank-2 transformations and only selected arcs. Transformations are written in one-line notation; an arrow labelled δ represents right multiplication by a selected generator $\delta \in D_2$.

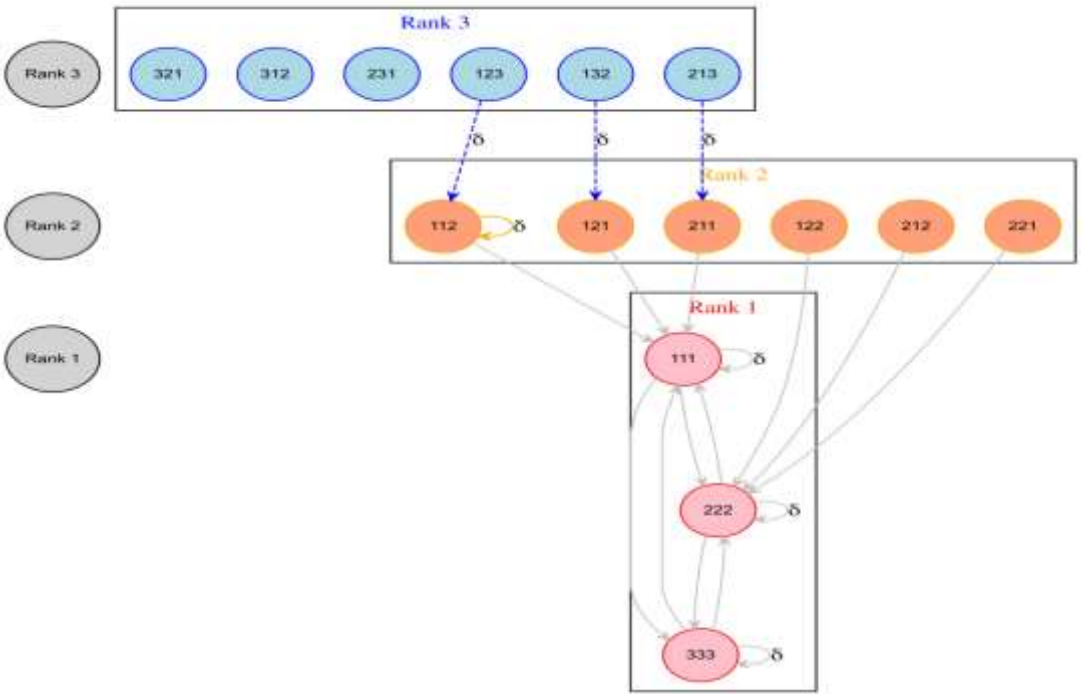


Figure 1: A selected labelled sub-digraph of $\Gamma_{3,2}$ = Cay(T_3, D_2). The diagram shows representative rank-2 vertices and selected products between the rank layers; it is not the complete 27-vertex, 279-arc Cayley digraph

Corollary 3.1.2 permits exactly the layer transitions $3 \to 2$, $2 \to 2$, $2 \to 1$ and $1 \to 1$, summarized in Figure 2. In particular, arcs do not all decrease rank: rank-preserving arcs occur inside the R-classes of ranks 1 and 2.



Figure 2: Rank-transition quotient of $\Gamma_{3,2}$. A loop on a rank layer indicates the existence of rank-preserving arcs, not merely vertex loops

The out-degrees of rank-1, rank-2 and rank-3 vertices are respectively 3, 9 and 18. The strongly connected components consist of one rank-1 R-class of size 3, three rank-2 R-classes of size 6, and six singleton rank-3 components. The digraph is weakly connected but not locally cyclic.

CONCLUSION

Within the original problem of Cayley digraphs of T_n generated by a Green D-class, we have obtained a substantially broader theory than connectivity alone. The prescribed-rank extension lemma gives the exact arc relation and prevents the loss of information that occurs when an unrestricted right factor is used. It yields both sides of the degree structure: out-degree depends only on the initial rank, whereas in-degree depends on the complete block-size profile of the terminal kernel. Example 3.7 shows that vertices in the same D-class can therefore have different in-degrees.

The graph separates naturally into a source layer and a terminal rank ideal. All transformations of rank greater than r have in-degree zero, every arc enters I_r, and every directed cycle lies in this core. Inside the core, the strongly connected components are precisely the Green R-classes. Contracting them produces an acyclic condensation whose maximum directed path length is exactly r when $r < n$. The exact distance formula to the constant class quantifies the rate at which right multiplication by D_r collapses rank and strengthens the earlier qualitative weak-connectivity result.

These results establish degree-profile, source-core, component, distance and rigidity invariants for the rank-class Cayley digraphs. Further work may determine the complete in-degree distribution by integer-partition type, exact undirected diameters, automorphism groups, spectra of the condensation and rank-quotient operators, and extensions to partial transformation and symmetric inverse semigroups.

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