

SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS: AN ADAPTIVE VARIATIONAL ITERATION APPROACH

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ABSTRACT

This work presents the study of Adaptive variational iteration method (AVIM), an enhanced semi-analytical technique for solving nonlinear partial differential equations (PDEs). The AVIM is a modification of the classical variational iteration method (VIM), in which an adaptive Lagrange multiplier is dynamically updated at each iteration based on the residual of the preceding approximation. This adaptive mechanism ensures improved convergence behavior and solution accuracy. The method was successfully applied to heat and Burgers's equation. In each case, iterations were computed and for the first three equations, the results of the iteration solutions were compared against He's Variational Iteration method and known exact solutions. Numerical comparisons were presented in tabular form, and corresponding graphs were plotted to visualize the convergence and accuracy of the method. Results indicate that AVIM yields highly accurate approximations in fewer iterations than the standard VIM. Furthermore, convergence graphs of successive iterations clearly demonstrate the superior performance of AVIM in capturing the solution behavior of nonlinear equations. The findings validate AVIM as a good and efficient tool for solving nonlinear PDEs.

Keywords: Adaptive variational iteration method, Variational iteration method, Heat equation, Burger's equation, Convergence

INTRODUCTION

Variational iteration methods are a powerful set of techniques used for solving differential equations indirectly. These methods center around formulating a functional, an expression that depends on the solution of the differential equation and potentially its derivatives. The core idea lies in minimizing or maximizing this functional subject to certain constraints, ultimately leading to the original differential equation.

Lagrange multipliers, a cornerstone of optimization theory, deal with constraints within the variational framework. By introducing Lagrange multipliers for each constraint, additional terms are incorporated into the functional. The solution is then obtained by finding the functions that minimize or maximize the modified functional while satisfying the imposed constraints.

Although Lagrange multipliers are not directly applied to many nonlinear partial differential equations, they serve as a valuable tool within variational methods. This approach becomes particularly advantageous for tackling complex nonlinear telegraph equations or those with intricate boundary conditions. We intend to extend the existing modified method to an adaptive method and use the method to solve Telegraph equations, Burgers' equations and other nonlinear Partial differential equations, which will serve as an interesting tool in the variational iteration method. This will complement other methods that have been used by other researchers. Solving nonlinear partial differential equations has been approached by various variational methods, notably the He's variational iteration method (He, 2000), including the Adomain method (Abbasbandy, 2019; Jiwari, 2012; Pandit *et al.*, 2015). Despite the potential of all the

methods, it is known that Lagrange multipliers in the variational iteration method reduce the successive application of the integral operator, minimizing the computational work to a tangible level while still maintaining a very high level of accuracy; hence, it is a clear advantage of this technique over the decomposition method (He, 2000). During the past three decades, different researchers have applied this method to a variety of linear and nonlinear Ordinary differential equations and partial differential equations, as discussed in Noor and Mohyud-Din (2007).

Mohyud-Din *et al.* (2009) solved telegraph equations by the variational iteration method using an approximate Lagrange multiplier. Pandit *et al.* (2015) calculated the solution of the second-order hyperbolic telegraph equation with variable coefficients. Jiwari *et al.* (2012) used the differential quadrature method to construct Sine-Gordon solutions of different types of telegraph equations. They used Lagrange interpolation and modified cubic B-spline differential quadrature methods for solving hyperbolic partial differential equations. Mittal and Jiwari (2011) found the numerical solutions of hyperbolic partial differential equations and quasilinear partial differential equations. Koksai (2011) used an operator difference method for the telegraph equation arising in transmission lines. Using the VIM, we can find the exact solution of the problem. Wazwaz (2007) said that in the variational iteration method (VIM), the linear and nonlinear structures are handled in a similar manner without any need for restrictive assumptions. The reliability of the method and the reduction in the size of the computational domain give this method a wider applicability.

Wazwaz (2007) employed VIM to present an analytic framework to handle the linear and the nonlinear Goursat problems. Ganji (2007) used He's variational iteration method to find solutions to the nonlinear Jaulent-Miodek equation, Korteweg-de Vries (KdV), and modified Korteweg-de Vries (MKdV) equations. Mohyud-Din *et al.* (2009) used the variational iteration method to solve telegraph equations; the method provides approximate solutions to the exact solutions. Dehghan and Shakari (2008a) solved a parabolic integrodifferential equation which describes problems in heat conduction in materials with memory. They applied this scheme to solve the Lane-Emden equation, which arises in astronomy. Dehghan and Shakari (2008b) employed the variational iteration procedure to solve the Cauchy reaction diffusion. Singh *et al.* (2017) proposed an analytical technique for the solution of Lane-Emden equations using the Adomian polynomial (Adomian, 1994), based on the VIM and control parameter. Noor and Mohyud-Din (2007) used variational iteration techniques to solve higher-order boundary value problems. The method provides an approximate solution to the original solution of the problem.

Das *et al.* (2016) proposed an algorithm-based method for numerically solving the Bratu-type and the Lane-Emden equations, where the recursive schemes for approximate solutions are calculated using the given boundary conditions. Al-Hayani (2022) developed a method using a combination of the variational iteration method and the homotopy analysis method to find the

approximate solution to a stiff initial-value problem (ivp) for ODEs by using a sequence of subintervals and the step size. Ojobor and Obihia (2021) developed a modified variational iteration method for solving nonlinear partial differential equations; their method uses the Adomian polynomial. The method is used to solve Cauchy problems of nonlinear partial differential equations. Their method was observed to perform better than the conventional variational iteration methods in terms of accuracy. It is observed that the adaptive method has not been adequately examined. Despite various modifications of the variational iteration methods, none have employed the adaptive modified method. Following the work by Noor and Mohyud-Din (2007), Ojobor and Obihia (2021). Interesting lecture notes on Variational Iteration Methods are Byrne (2009), and Cherkhaev, and Cherkhaev (2003). This research aims to extend these methods further by developing a new adaptive modified variational iteration method.

The following are the variants of the method, after which the problem formulation and methodology are discussed.

Variants of Variational Iteration Method

Variational iteration method

He's variational iteration method (VIM) is of the form

$$Lu + Nu = g(x) \quad (1)$$

where L is the linear operator, N , a nonlinear operator and $g(x)$ is the inhomogeneous term.

The correction functional is given by

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)(L(u_n(s)) + N(u_n(s)) - g(s))ds \quad (2)$$

Here, λ is referred to as the Lagrange multiplier (He, 2000). The method employs the variational principle and Lagrange multipliers for constructing the correction functions.

Modified Variational Iteration method

Mohyud-Din *et al.* (2009) modified the He's variational iteration method by introducing a new iteration formula which improved the accuracy and convergence of the He's method. Their modification involved using a

generalized Lagrange multiplier, which allows for more flexibility in the iteration process. They proposed the following modified variational iteration method in equation (1). Given any nonlinear differential equation of the form;

$$L(u) + N(u) = f(x)$$

where L is a linear operator, N is a nonlinear operator and $f(x)$ is a forcing term. The modified variational iteration formula is then given by

$$u_{n+1} = u_n + \int_0^x \lambda(\xi)L(u_n(\xi)) + N(u_n(\xi)) - f(\xi)d\xi \quad (3)$$

where $\lambda(\xi)$ is the generalized Lagrange multiplier and u_n is the n^{th} approximation of the solution. What differentiates their method from He's (2000) variational iteration method is a generalized Lagrange multiplier, defined as $\lambda = \xi - x$, in contrast to He's method.

While Ojobor *et al.* (2021) proposed that the correction functional is given by

$$u_{n+1}(x) = u_n + \int_{x_0}^x \lambda(\xi)R[u_n(\xi), \xi]d\xi + \beta\phi(u_n(x)) \quad (4)$$

where $R = L(u_n(\xi)) + N(u_n(\xi)) - f(\xi)$; here R is called the residual of the iteration.

Problem Definition and Methodology

Adaptive variational iteration method

The following general non-linear equation

$$L(u) + N(u) = g(x) \quad (5)$$

where L is the linear operator, N is a nonlinear operator and $g(t)$ is the analytical function. The Adaptive variational iteration method uses the following correction functional for equation (1)

$$u_{n+1}(x) = u_n(x) + \int_{x_0}^x \lambda(\xi) [L(u_n(\xi)) + N(u_n(\xi)) - g(\xi)] d\xi \quad (6)$$

Where: u_n is the n^{th} approximate solution and $N[u_n(x)]$ is the restricted variation, i.e $\delta N[u_n] = 0$ and λ is a general Lagrange multiplier which can be identified optimally by variational theory. Introducing an adaptive Lagrange multiplier $\lambda_n(\xi)$, which is updated at each iteration by evaluating the Lagrange multiplier at each iteration stage recursively, gives

$$u_{n+1}(x) = u_n(x) + \int_{x_0}^x \lambda_n(\xi) R_n(\xi) d\xi \quad (7)$$

in (7)

$$R_n = [L(u_n(\xi)) + N(u_n(\xi)) - g(\xi)]$$

Equation (6) is the correction functional of the adaptive variational iteration method. The main work here is to introduce a condition to refine $\lambda_n(\xi)$ such that the residual R_n is minimized in a weighted sense, i.e

$$\min_{\lambda_n} \int_{x_0}^x \lambda_n(\xi) R_n(\xi)^2 d\xi \quad (8)$$

and to solve for $\lambda_n(\xi)$ optimally. The local truncation error $E_n(x) = u_{n+1}(x) - u_n(x)$ if $R_n(x)$ exceeds a predefined threshold, then λ_n is refined, or the initial guess is adjusted.

A stopping criterion based on residual or error $\|R_n(\xi)\| < \epsilon$, $\|E_n(x)\| < \epsilon$ is introduced,

where is the desired accuracy.

The following fundamental theorem on variation would be useful in the analysis of our search for the exact solution of the problem.

Lemma (Fundamental Lemma of Calculus of Variation)

Let $g(x)$ be continuous on $[a, b]$ and let $\eta(x)$ be an arbitrary function on $[a, b]$, where η, η^0, η^{00} are continuous and $\eta(a) = \eta(b) = 0$.

If $\int_a^b g(x)\eta(x)dx = 0 \quad \forall$ such $\eta(x)$, then $g(x) = 0$ on $[a, b]$.

Proof:

The proof is done by contradiction. Let c be an arbitrary real number; suppose that $g(x) > 0$ at $x = c$. Since $g(x)$ is continuous there is a neighbourhood $M = (\alpha_1, \alpha_2)$ of c , i.e $\alpha_1 < c < \alpha_2$, in which $g(x) > 0$. Consider

$$\eta(x) = \begin{cases} (x - \alpha_1)^3(\alpha_2 - x)^3, & \alpha_1 < x < \alpha_2 \\ 0, & \text{elsewhere} \end{cases}$$

Then $g(x)\eta(x)$ has the same sign of $g(x)$, but $\int_a^b g(x)\eta(x)dx > 0$

Which is a contradiction.

The Euler-Lagrange Equation is stated below using the Fundamental Lemma in Calculus of variations.

Theorem (Euler-Lagrange Equation)

Suppose that

$$\eta(y) = \int_a^b \xi(x, y, y') dx,$$

where $a, b, y(a), y(b)$ are given, $y^{00}(x)$ is continuous and ξ is twice continuously differentiable function, then the extremal function $y = y_0(x)$ satisfies the equation $\frac{d}{dy} \left(\frac{\partial \xi}{\partial y'} \right) - \frac{\partial \xi}{\partial y} = 0$.

Proof:

This proof can be found in Ortega and Rheinboldt (1987). Consider the functional

$$\phi(\xi) = \int_a^b F(x, y, y') dx \quad (9)$$

Where $y = y_0 + \epsilon \eta(x)$, then $y' = y'_0 + \epsilon \eta'$, and let y_0 and η be satisfied. Hence ϕ is a function of with $\eta(a) = \eta(b) = 0$ differentiating (9) with respect to gives

$$\begin{aligned} \frac{d\phi}{d\epsilon} &= \int_a^b \left[\frac{\partial F}{\partial y} \frac{\partial \phi}{\partial \epsilon} + \frac{\partial F}{\partial y'} \frac{\partial y'}{\partial \epsilon} \right] dx \\ &= \int_a^b [f_y \eta(x) + F_{y'} \eta'(x)] dx \end{aligned} \quad (10)$$

Integrating the second term of (10) by parts gives

$$\begin{aligned} \int_a^b F_{y'} \eta' dx &= \eta(x) F_{y'} \Big|_a^b - \int_a^b \eta(x) \frac{d}{dx} F_{y'} dx \\ &= 0 - \int_a^b \eta(x) \frac{d}{dx} F_{y'} dx \\ &= - \int_a^b \eta(x) \frac{d}{dx} F_{y'} dx \end{aligned} \quad (11)$$

Since at $y = y_0(x)$ we have an extremal value then

$$\frac{d\phi}{d\epsilon} \Big|_{\epsilon=0} = 0 \quad (12)$$

Therefore, we have

$$\frac{d\phi}{d\epsilon} \Big|_{\epsilon=0} = \int_a^b [F_y \eta(x) + F_{y'} \eta'(x)] dx$$

$$0 = \int_a^b [F_y \eta(x) - \eta(x) \frac{d(F_{y'})}{dx}] dx$$

$$\int_a^b \eta(x) [F_y - \frac{d(F_{y'})}{dx}] dx \quad (13)$$

this implies

By fundamental lemma of Calculus of variation we have

$$\frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} = 0 \quad (14)$$

Convergence Analysis of Adaptive Variational Iteration Method (AVIM)

For the convergence analysis of our method, following equation (5) and (6), we make the following assumptions:

1. The operator L is bounded and linear.
2. The nonlinear operator N is Lipschitz continuous, i.e.,

$$\|N(u) - N(v)\| \leq K\|u - v\| \quad (15)$$

for some constant $K > 0$.

3. The adaptive multiplier satisfies

$$\|\lambda_n\| \leq M$$

for some constant $M > 0$.

Proof:

From (7), we have that

$$u_{n+1} - u_n = \int_{x_0}^x \lambda_n(\xi) R_n(\xi) d\xi. \quad (16)$$

Taking norms of (16) gives,

$$\|u_{n+1} - u_n\| \leq \int_{x_0}^x |\lambda_n(\xi)| \|R_n(\xi)\| d\xi.$$

Now,

$$R_n - R_{n-1} = L(u_n - u_{n-1}) + N(u_n) - N(u_{n-1}). \quad (17)$$

Using boundedness of L and Lipschitz continuity of N , $\|R_n - R_{n-1}\| \leq (K\|L\| + K)\|u_n - u_{n-1}\|$.

Using boundedness of λ_n ,

$$\| \lambda_n R_n - \lambda_{n-1} R_{n-1} \| \leq M(K\|L\| + K)\|u_n - u_{n-1}\|.$$

Hence,

$$\|u_{n+1} - u_n\| \leq (1 + M(K\|L\| + K))\|u_n - u_{n-1}\|, \quad (18)$$

where $\| \cdot \| = \| \cdot \|_{x=x_0}$.

If

$$M(K\|L\| + K) < 1, \quad (19)$$

Then the iteration becomes contractive and

$$\lim_{n \rightarrow \infty} \|u_{n+1} - u_n\| = 0.$$

Thus $\{u_n\}$ is a Cauchy sequence in the Banach space, and therefore converges to some u . Taking limits in the residual expression,

$$\lim_{n \rightarrow \infty} R_n(x) = L(u(x)) + N(u(x)) - g(x) = 0.$$

Therefore,

$$L(u) + N(u) = g(x),$$

which proves that the limit function u is the exact solution.

Numerical Experiments

The Adaptive Variational Iteration Method (AVIM) to solve some equations, such as the Heat and Burgers Equations. The approximate solutions obtained are compared with their corresponding exact solutions to

verify the accuracy and efficiency of the method. Comparison graphs are generated using MATLAB to validate the reliability and convergence behavior of the method.

Example 4.1

Consider the classical heat equation:

$$u_t = u_{xx}, \quad x \in [0, 1], \quad t > 0 \quad (20)$$

with the initial condition:

$$u(x, 0) = \sin(\pi x), \quad (21)$$

and boundary conditions:

$$u(0, t) = u(1, t) = 0. \quad (22)$$

The exact solution is:

$$u(x, t) = \sin(\pi x) e^{-\pi^2 t}. \quad (23)$$

Solution 4.1

Rewriting the Partial differential equation in the standard adaptive variational iteration form gives:

$$Lu + Nu = g(x, t),$$

where:

$$L(u) = \frac{\partial u}{\partial t}, \quad N(u) = -\frac{\partial^2 u}{\partial x^2}, \quad g(x, t) = 0 \quad (24)$$

Let the residual at the n -th iteration be:

$$R_n(x, \xi) = \frac{\partial u_n(x, \xi)}{\partial \xi} - \frac{\partial^2 u_n(x, \xi)}{\partial x^2} \quad (25)$$

The adaptive Lagrange multiplier is defined as:

$$\lambda_n(\xi) = \frac{1}{[R_n(x, \xi)]^2}.$$

The correction functional for our method is given by:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda_n(\xi) R_n(x, \xi) d\xi.$$

Given the initial approximation as:

$$u_0(x, t) = \sin(\pi x).$$

Then:

$$\frac{\partial u_0}{\partial \xi} = 0, \quad \frac{\partial^2 u_0}{\partial x^2} = -\pi^2 \sin(\pi x) \quad (26)$$

Thus,

$$R_0(x, \xi) = \pi^2 \sin(\pi x), \quad \lambda_0(\xi) = \frac{1}{\pi^4 \sin^2(\pi x)}.$$

Substituting into the correction functional:

$$u_1(x, t) = \sin(\pi x) + \int_0^t \frac{1}{\pi^4 \sin^2(\pi x)} \cdot \pi^2 \sin(\pi x) d\xi. \quad (27)$$

Simplifying:

$$u_1(x, t) = \sin(\pi x) + \frac{t}{\pi^2 \sin(\pi x)}.$$

Since the Laplacian eigenfunction $\sin(\pi x)$ satisfies:

$$\frac{\partial^2}{\partial x^2} \sin(\pi x) = -\pi^2 \sin(\pi x), \quad (28)$$

We obtain:

$$u_t = -\pi^2 \sin(\pi x).$$

So,

$$\begin{aligned} u_1(x, t) &= u_0(x) + \int_0^t u_{0t} d\tau = \sin(\pi x) - \pi^2 \sin(\pi x)t, \\ \Rightarrow u_1(x, t) &= \sin(\pi x)(1 - \pi^2 t). \end{aligned}$$

Now:

$$u_{1t} = -\pi^2 \sin(\pi x), \quad u_{1xx} = -\pi^2 \sin(\pi x)(1 - \pi^2 t),$$

So the residual becomes:

$$R_1(x, \xi) = -\pi^2 \sin(\pi x) + \pi^2 \sin(\pi x)(1 - \pi^2 \xi) = \pi^4 \xi \sin(\pi x).$$

Then,

$$u_2(x, t) = u_1(x, t) + \int_0^t \pi^4 \xi \sin(\pi x) d\xi = \sin(\pi x) \left(1 - \pi^2 t + \frac{(\pi^2 t)^2}{2!} \right). \quad (29)$$

Differentiating u_2 :

$$u_{2t} = -\pi^2 \sin(\pi x) + \pi^4 t \sin(\pi x), \quad u_{2xx} = -\pi^2 \sin(\pi x) \left(1 - \pi^2 t + \frac{(\pi^2 t)^2}{2} \right),$$

So,

$$R_2(x, t) = u_{2t} - u_{2xx} = \frac{(\pi^2 t)^2}{2} \pi^2 \sin(\pi x) = \frac{(\pi^2 t)^3}{2} \sin(\pi x).$$

Then,

$$\begin{aligned} u_3(x, t) &= u_2(x, t) + \int_0^t \frac{(\pi^2 \xi)^3}{2} \sin(\pi x) d\xi, \\ u_3(x, t) &= \sin(\pi x) \left(1 - \pi^2 t + \frac{(\pi^2 t)^2}{2!} - \frac{(\pi^2 t)^3}{3!} \right). \end{aligned}$$

Using induction, we obtain:

$$u_n(x, t) = \sin(\pi x) \sum_{k=0}^n \frac{(-\pi^2 t)^k}{k!},$$

and as $n \rightarrow \infty$, the series converges to:

$$u(x, t) = \sin(\pi x) e^{-\pi^2 t}.$$

Comparing our iteration solutions with the exact solutions at $t = 0.01, 0.02, 0.03, 0.04, 0.05$, we get the following results presented in Tables 1 – 5 and Figs 1 – 4.

Table 1: Comparison of AVIM Iterations and exact solution at $x = 0.5$

t	$[u_1]$	$[u_2]$	$[u_3]$	$[u_4]$	$[u_{\text{exact}}]$
0.00	1.0000	1.0000	1.0000	1.0000	1.0000
0.01	0.9013	0.9062	0.9061	0.9061	0.9060
0.02	0.8027	0.8222	0.8209	0.8209	0.8209
0.03	0.7039	0.7477	0.7434	0.7434	0.7437
0.04	0.6052	0.6831	0.6728	0.6738	0.6738
0.05	0.5065	0.6283	0.6083	0.6108	0.6105

Table 2: Comparison of AVIM and He's VIM solutions at $x = 0.5$ for various t values

$2 * t$	u_1		u_2		u_3		u_4	
	AVIM	VIM	AVIM	VIM	AVIM	VIM	AVIM	VIM
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.01	0.9013	0.9013	0.9062	0.8964	0.9061	0.8966	0.9061	0.8968
0.02	0.8026	0.8026	0.8222	0.7831	0.8209	0.7844	0.8209	0.7847
0.03	0.7039	0.7039	0.7477	0.6601	0.7434	0.6644	0.7434	0.6644
0.04	0.6052	0.6052	0.6831	0.5273	0.6728	0.5376	0.6738	0.5423
0.05	0.5065	0.5065	0.6283	0.3848	0.6083	0.4048	0.6108	0.4049

Table 3: Absolute errors of AVIM and He's VIM compared to exact solution at $x = 0.5$

$2 \times x$	$ u_1 - \text{Exact} $		$ u_2 - \text{Exact} $		$ u_3 - \text{Exact} $		$ u_4 - \text{Exact} $	
	AVIM	VIM	AVIM	VIM	AVIM	VIM	AVIM	VIM
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	-0.0047	-0.0047	0.0002	-0.0096	0.0.0001	-0.0094	0.0001	-0.0094
0.2	-0.0182	-0.0182	0.0013	-0.0378	0.0000	-0.0365	0.0000	-0.0365
0.3	-0.0398	-0.0398	0.0040	-0.00398	-0.0003	-0.0836	-0.0003	-0.0793
0.4	-0.0686	-0.0686	0.0093	-0.0686	-0.0010	-0.1465	-0.0010	-0.1362
0.5	-0.1040	-0.1040	0.0178	-0.2257	-0.0022	-0.2057	0.0000	-0.2034

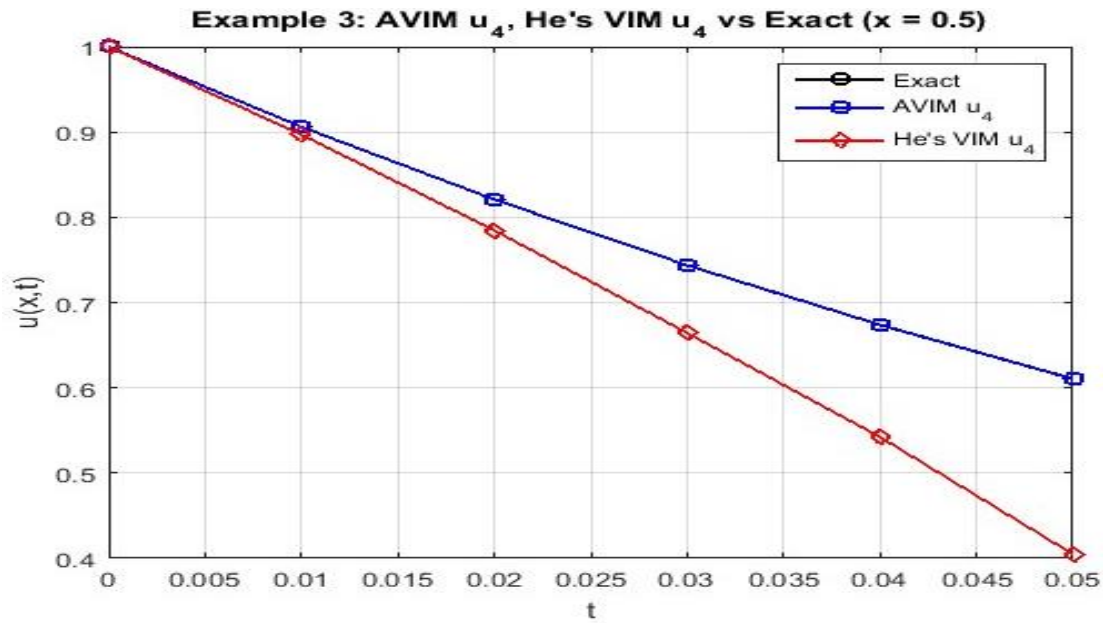


Figure 1: Comparison graph

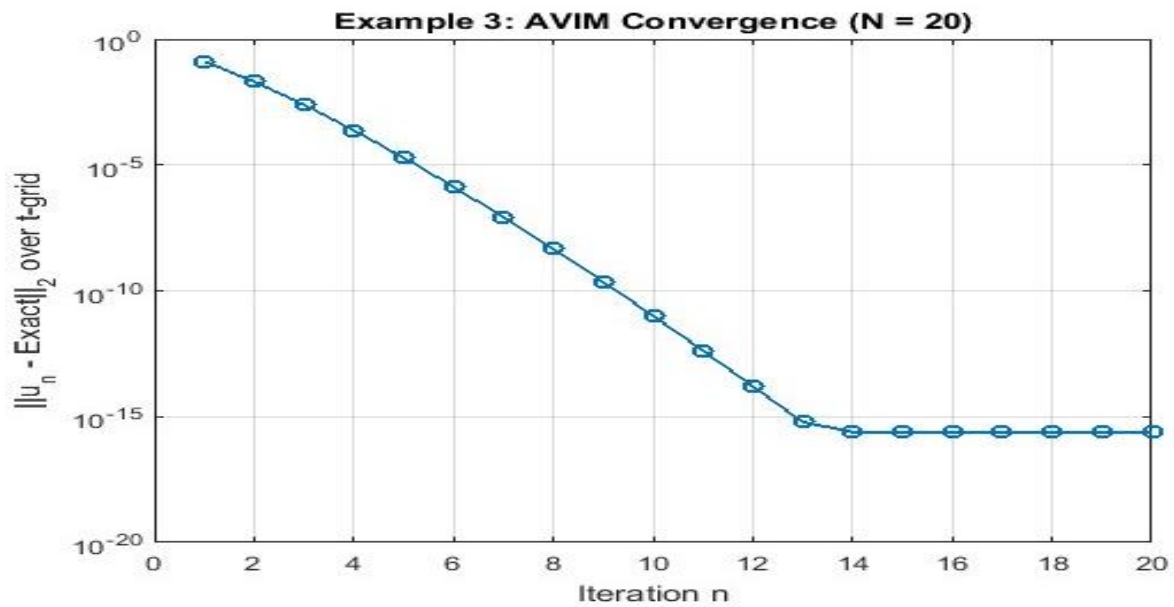


Figure 2: Convergence graph Example 4.2

Consider the following Burgers' equation

$$u_t + uu_x = 0 \quad (30)$$

subject to the following initial condition

$$u(x, 0) = -x \quad (31)$$

Using the correction functional

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda_n(s) R_n(s) ds \quad (32)$$

Solution 4.2 from the problem

$$R_n = u_n t + u_n u_{n,x},$$

and

$$\lambda_n = \frac{1}{R_n^2}$$

using the initial condition $u_0(x, t) = -x$ and $R_0 = u_{0,t} + u_0 u_{0,x}$ then on differentiating, we have $u_{0,t} = 0$, and $u_{0,x} = -1$ thus, $R_0 = (-x)(-1) = x$. Also, $\lambda_0 = \frac{1}{x^2}$. Substituting into the correction functional (34) gives

$$u_1(x, t) = u_0(x, t) + \int_0^t \frac{1}{x} ds$$

which on integration and solving gives

$$u_1(x, t) = -x + \frac{t}{x} \quad (33)$$

For the second iteration, we have

$R_1 = u_{1,t} + u_1 u_{1,x}$ differentiating $u_1(x, t)$ and substituting into R_1 we get

$$R_1 = \frac{1}{x} + \left(-x + \frac{t}{x}\right) \left(-1 - \frac{t}{x^2}\right)$$

Simplifying gives

$$u_3(x, t) = -x + \frac{t}{x} + \frac{x^2}{\sqrt{x^2 + 1}} \operatorname{artanh} \left(\frac{t}{x\sqrt{x^2 + 1}} \right) + \int_0^t \frac{1}{R_2(x, \xi)} d\xi.$$

Similarly,

$$R_3(x, t) = u_{3,t} + u_3 u_{3,x}$$

so that

$$u_4(x, t) = u_3(x, t) + \int_0^t \frac{1}{R_3(x, \xi)} d\xi.$$

$$R_1 = \frac{1}{x} + x - \frac{t^2}{x^3} \quad (34)$$

Therefore, $\lambda_1 = \frac{1}{R_1^2}$ so using the correction functional we have

$$u_2(x, t) = u_1(x, t) + \int_0^t \frac{1}{R_1^2} R_1 ds \quad (35)$$

On substituting in the integral gives

$$u_2(x, t) = -x + \frac{t}{x} + \int_0^t \frac{1}{\left(\frac{1}{x} + x - \frac{s^2}{x^3}\right)} d\xi \quad (36)$$

Recall that

$$R_1(x, t) = \frac{1}{x} + x - \frac{t^2}{x^3},$$

Let,

$$A = \frac{1}{x} + x, \quad B = \frac{1}{x^3},$$

Then from (36)

$$\begin{aligned} u_2(x, t) &= u_1(x, t) + \int_0^t \frac{d\xi}{A - B\xi^2}, \\ u_2(x, t) &= -x + \frac{t}{x} + \frac{1}{\sqrt{AB} \operatorname{artanh} \left(\xi \sqrt{\frac{B}{A}} \right)} \Big|_{\xi=0}^{\xi=t}, \\ &= -x + \frac{t}{x} + \frac{x^2}{\sqrt{x^2 + 1}} \operatorname{artanh} \left(\frac{t}{x\sqrt{x^2 + 1}} \right). \end{aligned}$$

Now,

$$R_2(x, t) = u_{2,t} + u_2 u_{2,x},$$

such that

Table 4: Comparison of AVIM iterations and exact solution at $x = 2$ for various t

t	Exact	u_0	u_1	u_2	u_3	u_4	$ u_4 - u_{\text{exact}} $
0.01	-1.9802	-2.0000	-1.9950	-1.9910	-1.9876	-1.9845	0.0043
0.02	-1.9608	-2.0000	-1.9900	-1.9820	-1.9751	-1.9689	0.0081
0.03	-1.9417	-2.0000	-1.9850	-1.9730	-1.9626	-1.9534	0.0111
0.04	-1.9231	-2.0000	-1.9800	-1.9640	-1.9502	-1.9378	0.0148
0.05	-1.9048	-2.0000	-1.9750	-1.9550	-1.9377	-1.9228	0.0175

Table 5: Comparison of AVIM and He's VIM with Exact solution at $x = 2$

t	Exact	AVIM (u_4)	$ u_4 - u_{\text{exact}} $	He's VIM (u_4)	$ u_4 - u_{\text{exact}} $
0.01	-1.9802	-1.9845	-0.0043	-2.0202	0.0400
0.02	-1.9608	-1.9689	-0.0081	-2.0408	0.0800
0.03	-1.9417	-1.9534	-0.0117	-2.0619	0.1201
0.04	-1.9231	-1.9378	-0.0147	-2.0834	0.1603
0.05	-1.9048	-1.9228	-0.0180	-2.1053	0.2005

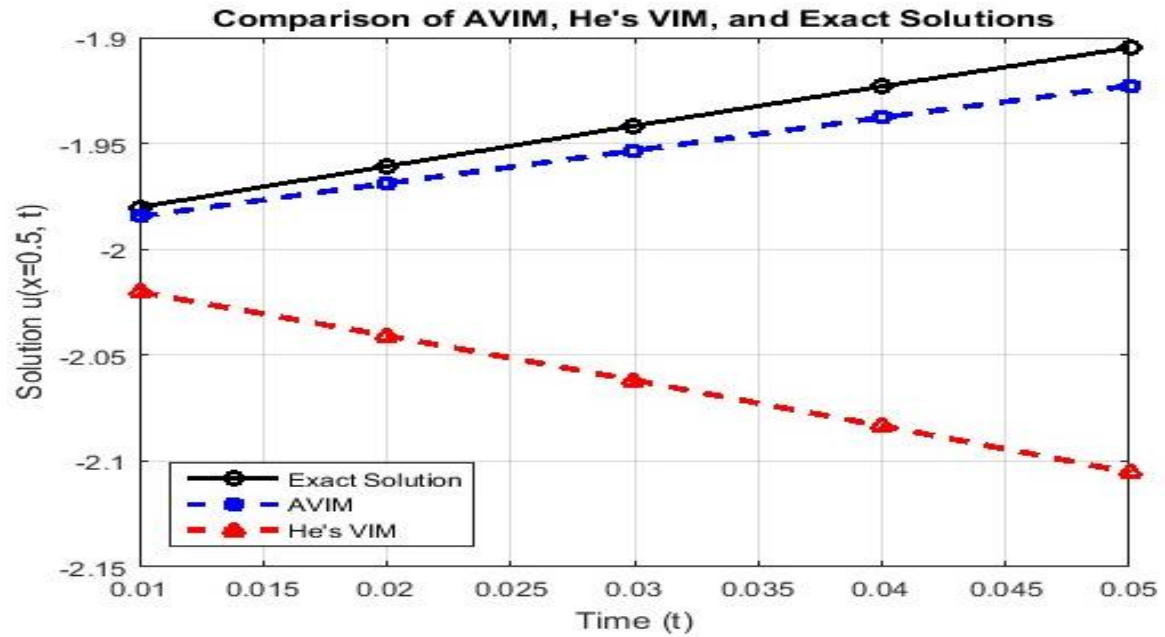


Figure 3: Comparison graph

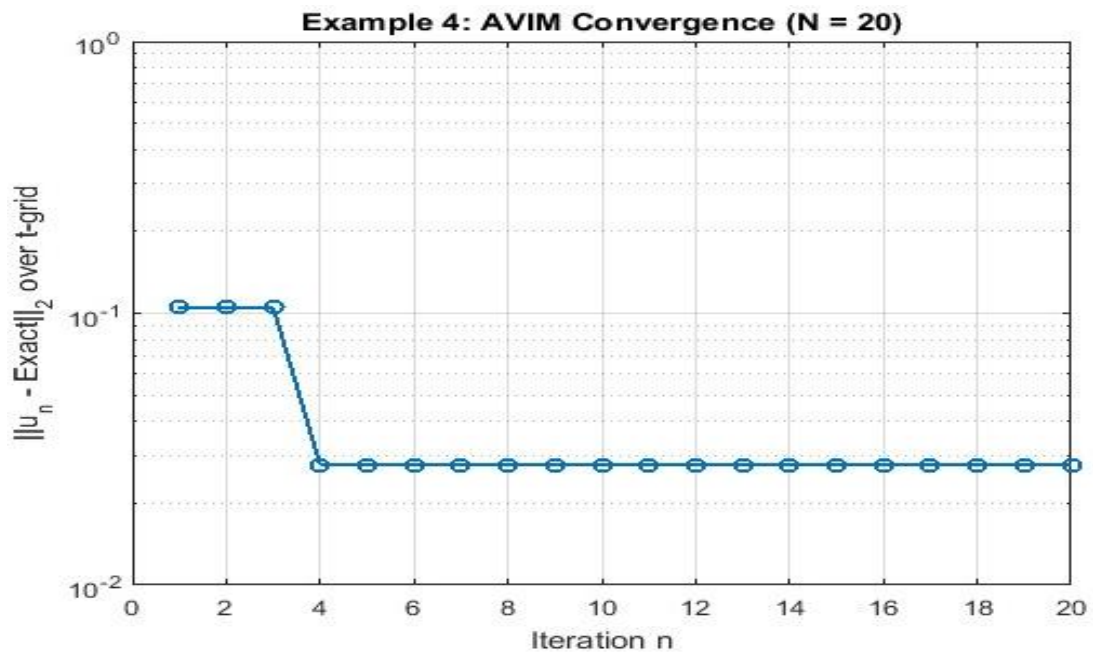


Figure 4: Convergence graph

DISCUSSION OF RESULTS

In Example 4.1 AVIM significantly outperforms He's VIM, as AVIM aligns very closely with the decaying exponential structure of the exact solution, whereas He's VIM produces larger errors, particularly as t increases. The stopping criterion is again achieved at the fourth iteration, where the truncated series representation matches the exact solution to very high accuracy, and the convergence graph exhibits a steep initial drop in error followed by exponential stabilization, clearly demonstrating the rapid convergence properties of AVIM for diffusion-type equations. Finally, in Example 4.2 AVIM proves to be the more effective method because it closely approximates the exact solution $u_{\text{exact}}(x, t) = -x/(1 + t)$, while

He's VIM produces noticeably larger errors and tends to overshoot the correct magnitude as seen in the tables of results. The stopping criterion is satisfied at the fourth iteration, where AVIM stabilizes, and further iterations do not significantly change the approximate solution, and the convergence graph reveals a monotonic decrease in the error norm with sharp improvement during the first three iterations followed by a plateau after the fourth, confirming the superior stability and accuracy of AVIM for nonlinear evolutionary PDEs.

CONCLUSION

This study presented a comprehensive investigation into the Variational Iteration Method (VIM) and its modifications, culminating in the development and implementation of a novel Adaptive Variational Iteration Method (AVIM). The method introduces a dynamic Lagrange multiplier that adapts to the residual at each iteration, thereby addressing the inefficiencies and slow convergence issues commonly associated with standard VIM and its variants. The AVIM was successfully applied to several important linear and nonlinear partial differential equations, including the Heat, and Burgers' equations. These are equations that frequently arise in physics, engineering, and applied mathematics, and often pose significant challenges due to their nonlinear nature. Through numerical experiments and comparative analysis, AVIM demonstrated improved performance by providing highly accurate approximations in fewer iterations. The convergence graphs and numerical tables showed that the method not only reduced computational effort but also maintained high reliability across different problem types.

Conflict of interest: The authors declare that there is no conflict of interest.

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