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PERTURBED CHEBYSHEV-FINITE DIFFERENCE METHOD FOR SOLVING THIRD-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT
Ordinary differential equations are crucial in modeling complex phenomena in fluid mechanics, structural engineering, and damping systems. However, obtaining analytical solutions remains challenging due to inherent nonlinearities and intricate boundary conditions. This paper proposes a Perturbed Chebyshev Finite Difference Method (PCFDM) specifically designed to solve third-order ordinary differential equations (ODEs). The proposed framework integrates Chebyshev basis polynomials with a Legendre-based perturbation terms to enhance the approximation space and minimize truncation errors. Numerical experiments on third-order boundary value problems associated with draining and coating flows reveal that the PCFDM significantly outperforms conventional finite difference method (FDM) in terms of accuracy and computational efficiency. The results confirm that the integration of orthogonal polynomial and perturbation provides a flexible and reliable approach for solving high-order differential equations.

Keywords: Ordinary differential equations, Chebyshev polynomials, Finite difference method, Perturbation, Legendre polynomials basis, Boundary value problems

INTRODUCTION
Third-order ordinary differential equations (ODEs) arise naturally in the mathematical modelling of many physical, engineering, and applied science problems (Coddington & Levinson, 1955). Unlike second-order models that typically describe equilibrium or oscillatory behaviour, third-order equations often characterise systems involving damping, jerk dynamics, or coupled transport processes (Ahmed, 2017). For example, certain boundary layer flow models in fluid mechanics are of third-order nonlinear differential equations, such as the classical Falkner-Skan equation (Falkner & Skan, 1931). Similarly, models describing viscoelastic beam behaviour, thin film flow, draining and Coating Flows, and feedback control mechanisms may be governed by third-order differential systems (Ahmed, 2017; Rao, 2007). In control engineering, higher-order actuator dynamics, motor position control, and suspension systems frequently lead to third-order differential models that describe the evolution of system states under feedback laws (Ogata, 2010).
Closed-form analytical solutions of third-order boundary value problems are generally difficult to obtain, particularly in the presence of nonlinearities, variable coefficients, or non-standard boundary conditions (Agarwal, 1986). Consequently, numerical methods have become indispensable tools for their investigation. Over the years, several numerical techniques have been developed for solving third-order ODEs, including finite difference methods, finite element methods, collocation methods, shooting techniques, and spectral methods (Ascher *et al*., 1995, LeVeque, 2007). Each of these approaches offers particular advantages depending on the structure of the problem and the desired accuracy.
Among these techniques, the finite difference method (FDM) remains one of the most widely used due to its conceptual simplicity and adaptability to various types of boundary conditions (LeVeque, 2007). In particular, the central difference approximation has been extensively employed for discretising derivatives because of its improved truncation error properties and second-order accuracy (Smith, 1985). The effectiveness of central difference scheme in solving boundary value problems is well established in the literature (Alvarez-Ramirez & Valdes-Parada, 2009; Endeshaw, 2019; Enemoh *et al*., 2025). The method provides discrete approximations by replacing continuous derivatives with algebraic difference quotients, thereby transforming differential equations into systems of linear or nonlinear algebraic equations. Despite its robustness, the classical finite difference approach produces solutions only at discrete grid points, often requiring an additional interpolation procedure to obtain a continuous representation of the solution (LeVeque, 2007). To address this limitation and enhance accuracy, spectral-based approximations using orthogonal polynomials have gained increasing attention (Amer *et al*., 2018; Ghimire *et al*., 2020; Mamadu & Njoseh, 2016; Cerdik & Mutlu, 2020; Yigit & Bayram, 2019). Chebyshev polynomials, in particular, are well known for their efficient approximation properties and rapid convergence for smooth functions (Mason & Handscomb, 2002). Their associated collocation points cluster near the endpoints of the interval, reducing Runge-type oscillations and improving stability.

In this study, we are motivated by the work of Taiwo and Fesojaye (2017), Uwaherena *et al*. (2020), Dung and Quang (2024), to enhance the classical central finite difference framework by integrating Chebyshev basis polynomials into the discretisation process and introducing a Legendre polynomial based perturbation. This hybrid Chebyshev finite difference approximation facilitates the derivation of continuous numerical solutions, thereby eliminating the need for an additional interpolation step to evaluate the solution at arbitrary points within the domain. The incorporation of Legendre polynomials basis perturbation further refines the approximation space and improves solution accuracy with minimal additional computational cost (Taiwo & Fesojaye, 2017).

Chebyshev Differential Equation and Polynomials Basis

The Chebyshev differential equation constitutes a special case of the Sturm–Liouville boundary value problem (Enemoh *et al*., 2025). In comparison with other interpolation bases, Chebyshev polynomials have been widely applied and are regarded as highly efficient and accurate for the approximation of polynomial functions. In comparison with other interpolation bases, Chebyshev polynomials have been widely applied and are regarded as highly efficient and accurate for the approximation of polynomial functions. The Chebyshev differential equation is given by Ishola *et al.* (2022):

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y(x) = 0 \qquad (1)$$

The solution of equation (1), which exhibits singularities at $x = [-1,1]$, can be represented as the Chebyshev polynomials basis (Enemoh *et al*., 2025).

Orthogonality Property of Chebyshev Polynomials

The polynomials $T_n(x)$ form complete orthogonal set on the interval $[1,-1]$ in $x$with the weighted function $w(x) = \frac{1}{\sqrt{1-x^2}}$ (Karjanto, 2002), such that

$$\int_{-1}^{1} w(x)T_n(x)T_m(x)dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n = 0 \\ \frac{\pi}{2}, & m = n \neq 0 \end{cases} \qquad (2)$$

Chebyshev Polynomial of First Kind

The Rodrigue's formula can be used to generate Chebyshev polynomials basis, $T_n(x)$

$$T_n(x) = \frac{(-2)^n n!}{(2n)!}\sqrt{1-x^2}\frac{d^n}{dx^n}(1-x^2)^{n-1/2}, \quad n = 0,1,2,3,\cdots \qquad (3)$$

Given that, the first two Chebyshev polynomials $T_0(x)$ and $T_1(x)$ are known, all other polynomials basis $T_n(x), n \geq 2$ can be generated by means of recurrence formula , defined as:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \; T_0(x) = 1, T_1(x) = 2x^2 - 1 \qquad (4)$$

Chebyshev Polynomial of Second Kind

Given that $x \in [-1,1]$ and $n = 1,2,\cdots N.$ The generating function of Chebyshev polynomials of second kind is given by (Karjanto, 2002):

$$\frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^2, \quad \forall\, |x| < 1, \quad |t| < 1 \qquad (5)$$

And the recurrence relation defined as (Karjanto, 2002):

$$U_0(x) = 1 \qquad U_1(x) = 2x$$
$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \; \forall \geq 1 \qquad (6)$$

Shifted Chebyshev Polynomials

For analytical and numerical investigations, it is frequently advantageous to transform the domain of the Chebyshev polynomials from the standard interval $[-1,1]$ to the shifted interval $0 \leq x \leq 1$ (Enemoh *et al*., 2025). This transformation is especially beneficial when dealing with boundary value problems formulated exclusively on the half interval $[0,1]$. Therefore, the shifted Chebyshev polynomials bases are obtained by Taiwo and Fesojaye (2017):

$$T_n^*(x) = (2x-1)T_n^*(x), \quad T_0^* = 1, T_1^*(x) = 2x - 1, \; n = 2,3\cdots N \qquad (7)$$

Legendre Differential Equation and Polynomials Basis

Legendre differential equation is of the form (Abdulkadir, 2019):

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \;\; n > 0, x < 1 \qquad (8)$$

The general solution to equation (8) is given as a sum of Legendre functions defined as follows (Abdulkadir, 2019):

$$y(x) = AP_n(x) + BQ_n(x), x < 1 \qquad (9)$$

Where $P_n(x)$ is the Legendre functions of the first kind and $Q_n(x)$ is the Legendre functions of the second kind. The functions $P_n(x)$ and $Q_n(x)$ are respectively defined by

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$Q_n(x) = \frac{1}{2}P_n(x)\ln\left(\frac{1+x}{1-x}\right)$$

Orthogonality of Legendre Polynomials

The Legendre polynomials $P_m(x)$ and $P_n(x)$ defined by Bos *et al*. (2017), is said to be orthogonal in the interval $[-1,1]$, provided:

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0, \quad \text{for } m \neq n \qquad (10)$$

Equation (10) can also be obtained as:

$$\int_{-1}^{1} [P_m(x)P_n(x)]^2 dx = \frac{2}{2n+1}, \quad \text{if } m = n \qquad (11)$$

They form a complete orthogonal basis for square-integrable functions on [-1, 1] and are widely used in approximation theory and numerical analysis.

Shifted Legendre Polynomials Basis

The solution of equation (8) is called Legendre functions of degree n. When n is a nonnegative integer, the Legendre functions are often referred to Legendre polynomials basis $P_n(x)$ (Al-Humedi & Al-Saadawi, 2021). The Legendre polynomials basis are defined on the interval [-1,1] and can be determined using the recurrence formulae,

$$L_0(z) = 1, L_1(z) = z,$$
$$L_{r+1}(z) = \frac{2r+1}{r+1}(z)L_r(z) - \frac{r}{r+1}L_{r-1}(z); r = 1,2,3 \quad (12)$$

The Legendre polynomial is one of the special orthogonal polynomials define on the interval $[-1,1]$ along side Chebyshev polynomial basis. In practice, the interval $[-1,1]$ is not too convenient for many problems (Al-Humedi & Al-Saadawi, 2021; Yaslan & Mutlu, 2020). To use this polynomial on the interval [0,1], hence the need for the Shifted interval. The shifted Legendre polynomials on the interval [0,1] is defined by introducing $z = 2x - 1$ in (14). Let the shifted Legendre polynomial $L_r(2x-1)$be denoted by $P_r(x)$and $P_r(x)$ can be obtained as follows

$$P_0(x) = 1, P_1(x) = 2x - 1$$
$$P_{r-1}(x) = \frac{2r+1}{r+1}(2x-1)P_r(x) - \frac{r}{r+1}P_{r-1}(x), x = 1,2,3 \quad (13)$$

Equation (15) with the initial condition is called the shifted Legendre polynomials generating function (Erjaee *et al*., 2013).

MATERIALS AND METHODS

Third-Order Boundary Value Problem

The general form of third-order boundary value problem of ordinary differential equations is given by:

$$A(x)y''' + B(x)y'' + C(x)y' + D(x)y = f(x) \quad (14)$$

Subject to the boundary conditions:

$$y(a) = \alpha_0, y'(a) = \alpha_1, y'(b) = \alpha_2, \quad x \in [a,b]$$

Formulation of Chebyshev Finite Difference Method (CFDM)

The approach in this study is designed to harness the properties of Chebyshev Polynomial Basis in the application of finite difference method. The finite difference approximation in this section are written in terms of Chebyshev basis $a_r T_r(x)$, of a given degree n:

$$y_n(x) = \sum_{r=0}^{n} a_r T_r(x) \quad (15)$$

Where $T_r(x)$ is the basis polynomials and a_r is the coefficient of the basis to be determine. The Chebyshev basis terms are defined as follows:

$$y_i = \left[\sum_{r=0}^{n} a_r T_r(x)\right]_i$$
$$y_{i+1} = \left[\sum_{r=0}^{n} a_r T_r(x)\right]_{i+1}$$
$$y_{i-1} = \left[\sum_{r=0}^{n} a_r T_r(x)\right]_{i-1}$$
$$y_{i-2} = \left[\sum_{r=0}^{n} a_r T_r(x)\right]_{i-2}$$

The following Chebyshev basis for central difference approximation of derivatives are derived:

$$y'(x_i) = \frac{1}{2h}\left([\textstyle\sum_{r=0}^{n} a_r T_r(x)]_{i+1} - [\sum_{r=0}^{n} a_r T_r(x)]_{i-1}\right), \quad O(h^2) \quad (16)$$

$$y''(x_i) = \frac{1}{h^2}\left([\textstyle\sum_{r=0}^{n} a_r T_r(x)]_{i+1} - 2[\sum_{r=0}^{n} a_r T_r(x)]_i + [\sum_{r=0}^{n} a_r T_r(x)]_{i-1}\right), \quad O(h^2) \quad (17)$$

$$y'''(x_i) = \frac{1}{2h^3}\left([\textstyle\sum_{r=0}^{n} a_r T_r(x)]_{i+2} - 2[\sum_{r=0}^{n} a_r T_r(x)]_{i+1} + 2[\sum_{r=0}^{n} a_r T_r(x)]_{i-1} - [\sum_{r=0}^{n} a_r T_r(x)]_{i-2}\right), \quad O(h^2) \quad (18)$$

The Chebyshev basis approximation method of the third-order boundary value problem of equation (14) is obtained by substituting the approximation basis terms $y'(x), y''(x)$, and $y'''(x)$ in (16) to (18), into (14). Thus, the following are obtained:

$$y'(x_i) \approx \frac{1}{2h}\left([\textstyle\sum_{r=0}^{n} a_r T_r(x)]_{i+1} - [\sum_{r=0}^{n} a_r T_r(x)]_{i-1}\right), \quad (19)$$

$$y''(x_i) \approx \frac{1}{h^2}\left([\textstyle\sum_{r=0}^{n} a_r T_r(x)]_{i+1} - 2[\sum_{r=0}^{n} a_r T_r(x)]_i + [\sum_{r=0}^{n} a_r T_r(x)]_{i-1}\right), \quad (20)$$

$$y'''(x_i) \approx \frac{1}{2h^3}\left([\textstyle\sum_{r=0}^{n} a_r T_r(x)]_{i+2} - 2[\sum_{r=0}^{n} a_r T_r(x)]_{i+1} + 2[\sum_{r=0}^{n} a_r T_r(x)]_{i-1} - [\sum_{r=0}^{n} a_r T_r(x)]_{i-2}\right) \quad (21)$$

Substituting equation (19) to (21) into the third-order equation (14), we obtained the CFDM scheme given by:

$$\frac{A(x)}{2h^3}[\textstyle\sum_{r=0}^{n} a_r T_r(x)]_{i+2} + \left(\frac{B(x)}{h^2} - \frac{A(x)}{h^3} + \frac{C(x)}{2h}\right)[\sum_{r=0}^{n} a_r T_r(x)]_{i+1} - \left(\frac{2B(x)}{h^2} - D(x)\right)[\sum_{r=0}^{n} a_r T_r(x)]_i + \left(\frac{A(x)}{h^3} + \frac{B(x)}{h^2} - \frac{C(x)}{2h}\right)[\sum_{r=0}^{n} a_r T_r(x)]_{i-1} - \frac{A(x)}{2h^3}[\sum_{r=0}^{n} a_r T_r(x)]_{i-2} = f(x) \quad (22)$$

Perturbed CFDM for Third-order Ordinary Differential Equations

To perturb (22), we consider a two-step perturbation of an nth degree of Legendre basis approximation, given as:

$$L_\tau = \tau_1 L_{n-2}(x) + \tau_2 L_{n-3}(x) = \sum_{k=1}^{n} \tau_k \, L_{n-k-1}(x), \; k = 1,2 \quad (23)$$

Thus, the perturbed Chebyshev basis approximation is obtained by:

$$\frac{A(x)}{2h^3}[\sum_{r=0}^{n} a_r \, T_r(x)]_{i+2} + (\frac{B(x)}{h^2} - \frac{A(x)}{h^3} + \frac{C(x)}{2h})[\sum_{r=0}^{n} a_r \, T_r(x)]_{i+1} - (\frac{2B(x)}{h^2} - D(x))[\sum_{r=0}^{n} a_r \, T_r(x)]_i + \left(\frac{A(x)}{h^3} + \frac{B(x)}{h^2} - \frac{C(x)}{2h}\right)[\sum_{r=0}^{n} a_r \, T_r(x)]_{i-1} - \frac{A(x)}{2h^3}[\sum_{r=0}^{n} a_r \, T_r(x)]_{i-2} = f(x) + L_r \quad (24)$$

The equation (24) can be expressed in the form:

$$\mathrm{M}\Phi = \mathrm{f} + P_\tau \quad (25)$$

Where the matrix coefficients, M is given by:

$$M = [m_{i+2}, \; m_{i+1}, \; m_i, \; m_{i-1}, \; m_{i-2}]$$

$$m_{i+2} = \frac{A(x_{i+2})}{2h^3},$$

$$m_{i+1} = \left(\frac{B(x_{i+1})}{h^2} - \frac{A(x_{i+1})}{h^3} + \frac{C(x_{i+1})}{2h}\right),$$

$$m_i = -\frac{2B(x_i)}{h^2} + D(x_i),$$

$$m_{i-1} = (\frac{A(x_{i-1})}{h^3} + \frac{B(x_{i-1})}{h^2} - \frac{C(x_{i-1})}{2h}),$$

$$m_{i-2} = \frac{A(x_{i-2})}{2h^3}$$

and

$$\Phi = \begin{pmatrix} [a^T T(x)]_{i+2} \\ [a^T T(x)]_{i+1} \\ [a^T T(x)]_i \\ [a^T T(x)]_{i-1} \\ [a^T T(x)]_{i-2} \end{pmatrix}$$

Such that,

$$a^T T(x) = a_0 T_0(x) + a_1 T_1(x) + a_2 T_2(x) + a_3 T_3(x) + a_4 T_4(x) + \cdots + a_n T_n(x) \quad (26)$$

Thus, $\Phi = \mathbf{aT}$ represents the dot product of the vector **a** with the matrix **T** (where **T** has the Chebyshev polynomial basis evaluated at different points as its rows). And nonhomogeneous part f evaluated at x is given by

$$f = (f(x_i)) \quad (27)$$

Remark: The given problem is homogeneous whenever $f = 0$ and the perturbed method derived in equation (24) tends to CFDM if the perturbed term $L_\tau = 0$. To determine the adjustable parameters $a_0, a_1, a_2, \dots, a_{n-2}$ with the perturbation parameters τ_1 and τ_2, an $n-3$ algebraic equations are generated using equation (25) at $x_i \in [a,b]$ and the other three algebraic equations are generated using the boundary conditions for nth Chebyshev basis polynomials given by

$$y(x) = \sum_{r=0}^{n} a_r T_r(x) \quad (28)$$

The approximate series solution is obtained with determined adjustable parameters, a_r^* given by

$$y^*(x) = \sum_{r=0}^{n} a_r^* T_n(x) \quad (29)$$

Examples of Third Order Boundary Value Problems

Example 2.1: Consider the boundary value problem associated with draining and coating flows, given as:

$$y^{(3)} - xy = (x^3 - 2x^2 - 5x - 3)e^x, \; y(0) = 0, \quad y^{(1)}(0) = 1, \quad y^{(1)}(1) = -e$$

The analytical solution of this problem is

$$y(x) = x(1-x)e^x$$

Example 2.2: Consider the boundary value problem:

$$y^{(3)} + y = (7 - x^2)\cos x + (x^2 - 6x - 1)\sin x$$

subject to the boundary conditions

$$y(0) = 0, \quad y^{(1)}(0) = -1, \quad y^{(1)}(1) = 2\sin 1.$$

The analytical solution of this problem is given as:

$$y(x) = (x^2 - 1)\sin x.$$

RESULTS AND DISCUSSION

In this section, the proposed numerical scheme, PCFDM is simulated using MATLAB software to solve some third order boundary value problems. The results are compared to the conventional finite difference approximation and the absolute errors are computed.

The approximate series solution of Example 2.1 using PCFDM with Shifted Chebyshev basis polynomials of degree $n = 8$, is given by $y_{a^*}^*(x)$:

$$
\begin{aligned}
y_{a^*}^*(x) = {} & \frac{22344169875066203077e^{1/2}}{1434735106923190 5068}+\frac{68792756475024102 9989e^{1/4}}{26973020010155981 52784}+\frac{22640003436566693107e^{3/4}}{5738940427692762 0272} \\
& -\frac{43805295347735651 2685e^{3/8}}{5057441251904246 53647}-\frac{18649179174823688 3023e^{5/8}}{1685813750634748 84549} \\
& +(2x-1)[\frac{22976942701491546 0619e^{1/2}}{11477880855385524 0544}+\frac{81053763805999053 7757e^{1/4}}{26973020010155981 52784} \\
& \quad +\frac{29583428796945095 591e^{3/4}}{5738940427692762 0272}-\frac{55881407222021075 9329e^{3/8}}{5057441251904246 53647} \\
& \quad -\frac{24323634151355005 7251e^{5/8}}{1685813750634748 84549}-\frac{155616121861267700147751810817667 65391}{36442704860038623192684559394345 582592}] \\
& +(8x^2-8x+1)[\frac{49405130055124789 25e^{1/2}}{1434735106923190 5068}+\frac{45362099384132989 31e^{1/4}}{1685813750634748 84549} \\
& \quad +\frac{3554954968559167 54e^{3/4}}{358683776730797 6267}-\frac{95995997139325895 885e^{3/8}}{5057441251904246 53647} \\
& \quad -\frac{46053770335794442 999e^{5/8}}{1685813750634748 84549}-\frac{211086880732184865800844374438 5334429}{911067621500965579817113984858 6395648}] \\
& -(128x^4-256x^3+160x^2-32x+1)[\frac{2490491456319729 59e^{1/2}}{358683776730797 6267} \\
& \quad +\frac{17180854485857344 193e^{1/4}}{26973020010155981 52784}+\frac{10530760500032615 59e^{3/4}}{5738940427692762 0272} \\
& \quad -\frac{21139087570941849 665e^{3/8}}{5057441251904246 53647}-\frac{85149373112325994 03e^{5/8}}{1685813750634748 84549} \\
& \quad +\frac{144193966348457234894509288514 0581}{911067621500965579817113984858 6395648}] \\
& -(32x^3-48x^2+18x-1)[\frac{43438649892543394 29e^{1/2}}{2869470213846381 0136}+\frac{55028908055300943 423e^{1/4}}{26973020010155981 52784} \\
& \quad +\frac{20143664112785331 99e^{3/4}}{5738940427692762 0272}-\frac{12928499040886251 602e^{3/8}}{1685813750634748 84549} \\
& \quad -\frac{16888956391102083 414e^{5/8}}{1685813750634748 84549}+\frac{373394633109790893033876657769 345}{303689207166988526605704661619 5465216}] \\
& -(512x^5-1280x^4+1120x^3-400x^2+50x-1)[\frac{2935611496070369 647e^{1/2}}{11477880855385524 0544} \\
& \quad +\frac{63055493112456409 39e^{1/4}}{13486510005077990 76392}+\frac{2863532773859574 2e^{3/4}}{358683776730797 6267} \\
& \quad -\frac{83897839246512786 43e^{3/8}}{5057441251904246 53647}-\frac{36494293597452522 61e^{5/8}}{1685813750634748 84549} \\
& \quad +\frac{211082314890000830363489157565 9069}{36442704860038623192684559394345 582592}] \\
& -(2048x^6-6144x^5+6912x^4-3584x^3+840x^2-72x+1)[\frac{1061918528223235 09e^{1/2}}{1434735106923190 5068} \\
& \quad +\frac{10710947419953905 9e^{1/4}}{6743255002538995 38196}+\frac{4098954860226727 3e^{3/4}}{1434735106923190 5068} \\
& \quad -\frac{12710718728398330 25e^{3/8}}{5057441251904246 53647}-\frac{13326690100960050 43e^{5/8}}{1685813750634748 84549} \\
& \quad +\frac{342943042772502662791871269297 73}{455533810750482789908556992429 3197824}] \\
& -\frac{889835800312179573779429205892 289797}{455533810750482789908556992429 3197824}.
\end{aligned}
$$

Table 1: Example 2.1 ($n = 8$): Exact, FDM and PCFDM solutions with absolute errors (AEr)

x	Exact	FDM	FDM AEr	PCFDM	PCFDM AEr
0.000	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.125	0.1239381121	0.1355208477	0.0115827356	0.1243120011	0.0003738890
0.250	0.2407547656	0.2428812241	0.0021264585	0.2421133243	0.0013585587
0.375	0.3410136128	0.3551742304	0.0141606176	0.3438070568	0.0027934440
0.500	0.4121803177	0.4176785312	0.0054982135	0.4166832111	0.0045028934
0.625	0.4378701463	0.4566295982	0.0187594519	0.4441330033	0.0062628570
0.750	0.3969375031	0.4087708730	0.0118333699	0.4047551010	0.0078175979
0.875	0.2623769853	0.2898547523	0.0274777670	0.2712929820	0.0089159967
1.000	0.0000000000	0.0236925596	0.0236925596	-0.0093314954	0.0093314954
Max AEr			$2.7477766973 \times 10^{-2}$		$9.331495417152\times 10^{-3}$

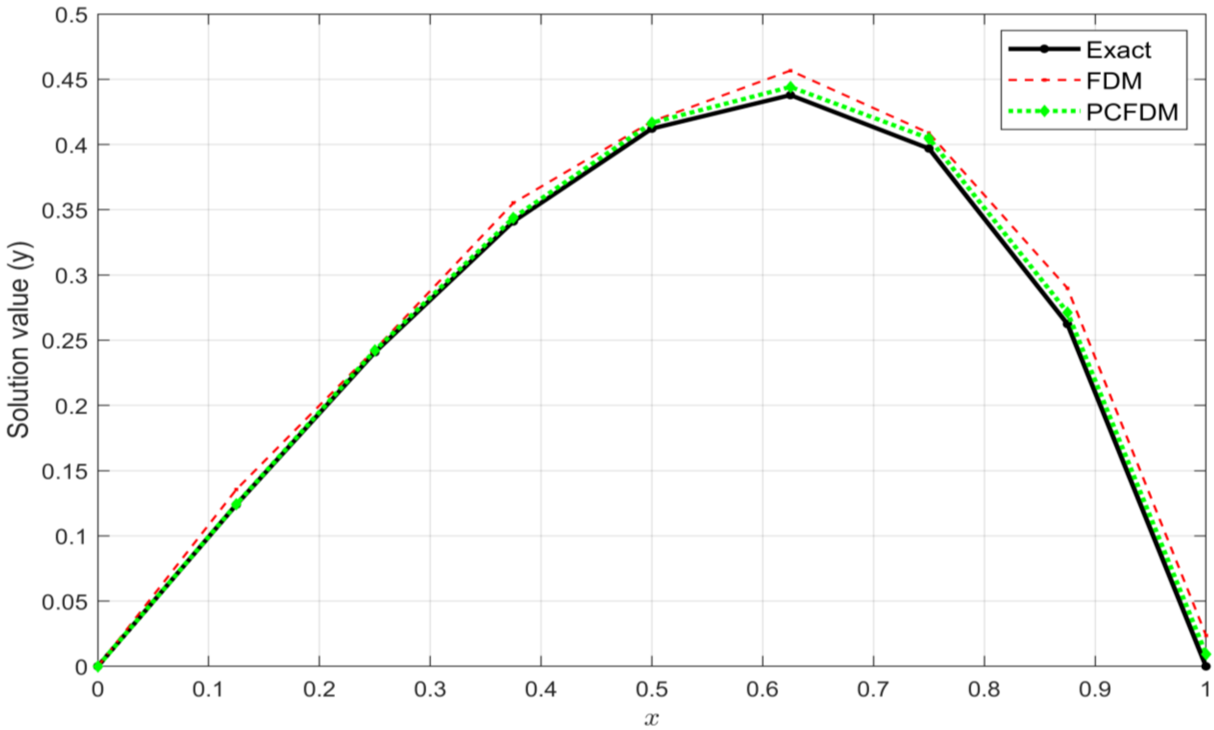

Figure 1: Graph of Example 2.1, comparison of solutions

The approximate series solution of Example 2.2, using PCFDM with Shifted Chebyshev basis polynomials of degree $n = 8$, is given by $y_{a^*}(x)$:

$$
\begin{aligned}
y_{a^*}(x) = \;& \frac{6814090515622562279\cos\left(\frac{3}{8}\right)}{5556781026733059342} - \frac{6368331768462801557\cos\left(\frac{1}{4}\right)}{14818082737954824912} \\
& - \frac{15950803750190248789\cos\left(\frac{3}{4}\right)}{44454248213864474736} - \frac{18625909649067891027\cos\left(\frac{1}{2}\right)}{9878721825303216608} \\
& + \frac{7136708717129555521\cos\left(\frac{5}{8}\right)}{617420114081451038} + \frac{10347727582815495015\sin\left(\frac{1}{2}\right)}{9878721825303216608} \\
& + \frac{2237521972703146493\sin\left(\frac{1}{4}\right)}{14818082737954824912} + \frac{12234111614223588877\sin\left(\frac{3}{4}\right)}{44454248213864474736} \\
& - \frac{3088847409131867639\sin\left(\frac{3}{8}\right)}{5556781026733059342} - \frac{470719085597906833\sin\left(\frac{5}{8}\right)}{617420114081451038} \\
& + (2x-1)\Big[\frac{8674027799023354837\cos\left(\frac{3}{8}\right)}{5556781026733059342} - \frac{7515162396239425661\cos\left(\frac{1}{4}\right)}{14818082737954824912} \\
& \quad - \frac{20778934906727162321\cos\left(\frac{3}{4}\right)}{44454248213864474736} - \frac{7463589408413374 50\cos\left(\frac{1}{2}\right)}{308710057040725519} \\
& \quad + \frac{9280362405159800 99\cos\left(\frac{5}{8}\right)}{617420114081451038} + \frac{414643856022965250\sin\left(\frac{1}{2}\right)}{308710057040725519} \\
& \quad + \frac{2640462463543581989\sin\left(\frac{1}{4}\right)}{14818082737954824912} + \frac{15937241336227629353\sin\left(\frac{3}{4}\right)}{44454248213864474736} \\
& \quad - \frac{3931962487484390917\sin\left(\frac{3}{8}\right)}{5556781026733059342} - \frac{612109009702029427\sin\left(\frac{5}{8}\right)}{617420114081451038} \\
& \quad + \frac{229858351607003103472315085567883}{139030649785413601247114205488 7424}\Big] \\
& + (8x^2-8x+1)\Big[\frac{1460337992508962351\cos\left(\frac{3}{8}\right)}{5556781026733059342} - \frac{3364070027295737 65\cos\left(\frac{1}{4}\right)}{7409041368977412456} \\
& \quad - \frac{9777267634018378 99\cos\left(\frac{3}{4}\right)}{11113562053466118684} - \frac{4028503775394319947\cos\left(\frac{1}{2}\right)}{9878721825303216608} \\
& \quad + \frac{1720484729316789 53\cos\left(\frac{5}{8}\right)}{617420114081451038} + \frac{2238057652996844415\sin\left(\frac{1}{2}\right)}{9878721825303216608} \\
& \quad + \frac{1181970550130934 85\sin\left(\frac{1}{4}\right)}{7409041368977412456} + \frac{7499069350363611 07\sin\left(\frac{3}{4}\right)}{11113562053466118684} \\
& \quad - \frac{6619755364676161 91\sin\left(\frac{3}{8}\right)}{5556781026733059342} - \frac{1134787800187669 69\sin\left(\frac{5}{8}\right)}{617420114081451038} \\
& \quad + \frac{2243759689408208394439921028437628 5}{1334694237939970571972296372691927 04}\Big] \\
& - (128x^4-256x^3+160x^2-32x+1)\Big[\frac{322878728451380819\cos\left(\frac{3}{8}\right)}{5556781026733059342} \\
& \quad - \frac{1598218271080707 53\cos\left(\frac{1}{4}\right)}{14818082737954824912} - \frac{7307482237757352 49\cos\left(\frac{3}{4}\right)}{44454248213864474736} \\
& \quad - \frac{8182635483773679 75\cos\left(\frac{1}{2}\right)}{9878721825303216608} + \frac{3211162819931179 7\cos\left(\frac{5}{8}\right)}{617420114081451038} \\
& \quad + \frac{4545908602096488 75\sin\left(\frac{1}{2}\right)}{9878721825303216608} + \frac{5615361492986269 7\sin\left(\frac{1}{4}\right)}{14818082737954824912} \\
& \quad + \frac{5604767929930396 57\sin\left(\frac{3}{4}\right)}{44454248213864474736} - \frac{1463618837399197 79\sin\left(\frac{3}{8}\right)}{5556781026733059342} \\
& \quad - \frac{2118001008890778 1\sin\left(\frac{5}{8}\right)}{617420114081451038} + \frac{142020908458119720437818842721 45}{1334694237939970571972296372691927 04}\Big] \\
& - (512x^5-1280x^4+1120x^3-400x^2+50x-1)(\cdots) \\
& - (2048x^6-6144x^5+6912x^4-3584x^3+840x^2-72x+1)(\cdots) \\
& - \frac{266413809959310015283939621044695}{1334694237939970571972296372691927 04}.
\end{aligned}
$$

Table 2: Example 2.2 ($n = 8$): Exact, FDM and PCFDM solutions with absolute errors (AEr)

x	Exact	FDM	FDM AEr	PCFDM	PCFDM AEr
0.000	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.125	-0.1227266907	-0.1207773631	0.0019493276	-0.1224730621	0.0002536286
0.250	-0.2319412118	-0.2271066192	0.0048345926	-0.2310061747	0.0009350371
0.375	-0.3147654547	-0.3080154849	0.0067499698	-0.3128731903	0.0018922644
0.500	-0.3595691540	-0.3502858962	0.0092832578	-0.3565960809	0.0029730730
0.625	-0.3565436507	-0.3469976680	0.0105459827	-0.3524999542	0.0040436965
0.750	-0.2982169575	-0.2860670715	0.0121498860	-0.2932360654	0.0049808921
0.875	-0.1798930083	-0.1676495043	0.0122435040	-0.1742385927	0.0056544156
1.000	0.0000000000	0.0124812378	0.0124812378	0.0059150109	0.0059150109
Max AEr			$1.2481237794 \times 10^{-2}$		$5.915010859382 \times 10^{-3}$

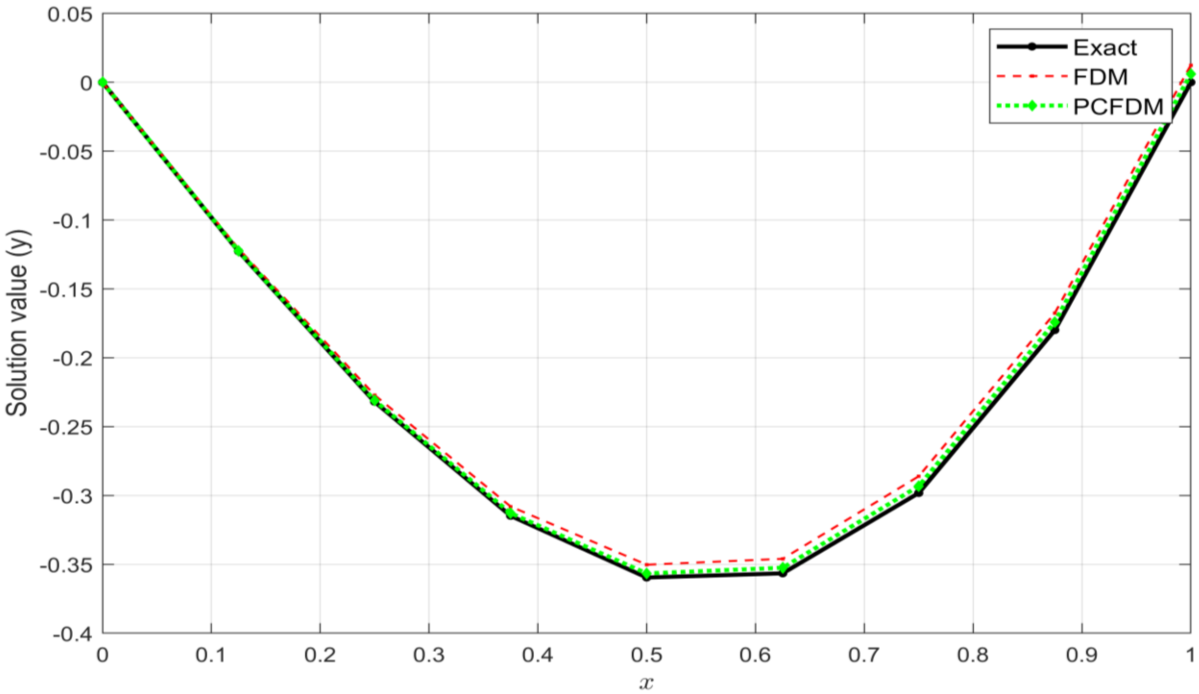

Figure 2: Graph of Example 2.2, comparison solutions

In this study, a perturbed Chebyshev finite difference method (PCFDM) was developed and applied to third-order boundary value problems arising in coating flow models. Numerical simulations, implemented in MATLAB, were compared with the conventional finite difference method (FDM) and the analytical solutions.

The results of Example 2.1, presented in Table 1, show that the computed absolute error (AEr) and its maximum for the PCFDM scheme, $9.331495417152 \times 10^{-3}$, are significantly lower than those of FDM, $2.7477766973 \times 10^{-2}$ and 1.64×10^{-2} reported by El-Danaf (2008).A similar improvement is observed in Example 2.2 (Table 2), where the maximum absolute error for PCFDM is $5.915010859382 \times 10^{-3}$ compared to $1.2481237794 \times 10^{-2}$ for the FDM and 8.8839×10^{-3} reported by El-Danaf (2008). The graphical comparisons in Figs 1 and 2 further confirm the closer agreement of the PCFDM results with the analytical solutions than those obtained using the conventional FDM.

These results demonstrate that the PCFDM is not only more accurate than conventional FDM but also maintains a continuous representation of the solution, making it a superior tool for engineering applications where derivative evaluation at arbitrary points is required. Hence, the method can be applied to a broad class of third-order initial and boundary value problems with a high degree of Chebyshev basis approximation.

CONCLUSION

In this paper, a perturbed Chebyshev finite difference method is formulated for solving third-order boundary value problems arising in draining and coating flow models. The method produces approximate series solution for the tested problems and shows excellent agreement with the analytical solutions. The numerical comparisons indicate improved accuracy over the classical finite difference method and earlier approaches, including that of El-Danaf (2008).

The proposed PCFDM framework is flexible and can be extended to other third-order models, including those describing travelling waves on thin liquid films under the influence of gravity and surface tension. Overall, the method provides an efficient and reliable tool for third-order ordinary differential equations.

Conflict of interest: The authors declare that there are no conflicts of interest associated with this study.

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