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PRODUCTS OF NILPOTENTS IN PARTIAL TRANSFORMATION SEMIGROUPS USING DIGRAPHIC PATHS AND CHAINS

U. M. Yusuf[1], A. M. Babayo[2] and M. A. Mbah[1]
[1]*Department of Mathematics, Federal University of Lafia, Nasarawa State, Nigeria*
[2]*Department of Mathematics and Statistics, Federal University of Kashere, Gombe State, Nigeria*

ABSTRACT
In this paper, we investigate the factorization of singular partial self-maps on a finite set into products of the least number of nilpotent elements. This research demonstrates that the semigroup of such maps can be expressed within a union of nilpotent-generated sets, specifically up to the third power. Some of our key findings include the determination of the nilpotent rank and the nilpotent depth for these maps, which vary based on whether the set size is even or odd. Additionally, this study surveys the relationship between these results and Stirling numbers, leveraging the Vagner Theorem and digraphic representations. We also examine stable quasi-idempotents, which correspond to specific digraphic paths and chains, providing further insights into the structure of partial transformation semigroups.

Keywords: depth formula, full transformation, idempotents, nilpotents, semigroups

INTRODUCTION

In semigroup theory, consideration of products of idempotent elements started from the work of J. M. Howie (Howie, 1966) who showed that, in the semigroup $\mathcal{T}_X(\text{or } T_X)$ of all full tranformations of a finite set X, every singular element (non-permutations) is a product of idempotents. That is if E denotes the set of idempotents in $\mathcal{T}_X\backslash\mathcal{S}_X$ and E_1 denotes the set of idempotents of defect one in $\mathcal{T}_X\backslash\mathcal{S}_X$, then $\mathcal{T}_X\backslash\mathcal{S}_X = \langle E\rangle = \langle E_1\rangle$.

For the case of infinite set X, Howie (1966) gave a description of the subsemigroup generated by idempotents of the full transformation semigroup. This results were also considered by other authors in the semigroup of matrices and transformations of other mathematical structures (Erdos, 1967; Howie, 1978) who characterized the minimum defect one idempotent required to generate $\mathcal{T}_X\backslash\mathcal{S}_X$using technique from graph theory, and showed that the cardinality of such a minimum generating set is $\frac{n(n-1)}{2}$, which is thus the idempotent rank of $\mathcal{T}_X\backslash\mathcal{S}_X$. For each element $\alpha \in \mathcal{T}_X\backslash\mathcal{S}_X$ the minimum number of idempotents in E_1 required to express α, that is the E_1-depth of α was obtained by Howie as$g(\alpha) = n + c(\alpha) - f(\alpha)$ (Howie, 1980).

More works on transformation semigroups followed the works of Howie. These included the works of Garba (1990) on the semigroup of all partial transformations of $\{1,2,\ldots,n\}$, Garba (1994a) and Garba (1994b) which extended the result on the semigroup of all partial one-one order-preserving maps on $X_n = \{1,2,\ldots,n\}$ and showed that the strictly partial transformations on the set $X_n = \{1,2,\ldots,n\}$ that is nilpotent-generated if n is even or odd has rank equal to $n+2$ respectively. Also, Yang (1998) considered the nilpotent ranks of the principal factors of certain semigroups of partial transformations.

Madu (1999) presented that an element α of an arbitratry semigroup S is called nilpotent if $\alpha^m = 0$ (the zero mapping) for some integer $m \geq 1$. The orders $|X_n| = n$, $|S_n| = n!$, $|T_n| = n^n$, $|P_n| = (n+1)^n$, $|B_n| = 2^{n^2}$. Madu (1999) also presented the Vagner Theorem: For each αin P_n, define the transformation α^* of $X_n^0 = X_n \cup \{0\}$ by $x\alpha^* = \begin{cases} x\alpha & \text{if } x \in dom(\alpha) \\ 0 & \text{if } x \notin dom(\alpha) \end{cases}$. Then α^* belongs to the subsemigroup P_n^* of U_n consisting of all those transformations of X_n^0 leaving 0 fixed. Conversely if $\beta \in P_n^*$, then its restriction to X_n as $\beta|_{X_n} = \beta \cap (X_n \times X_n)$ is a partial transformation of X_n. The domain of $\beta|_{X_n}$ is the set of all of all x in X_n for which $x\beta \neq 0$. Then the mapping $\alpha \to \alpha^*$ and $\beta \to \beta|_{X_n}$ are mutually inverse isomorphisms of P_n onto P_n^* and vice-versa since if $\alpha_1, \alpha_2 \in P_n$ and $x \in dom\ (\alpha_1)$, then $x\alpha_1\alpha_2 = (x\alpha_1)\alpha_2 = (x\alpha_1)\alpha_2^*$ (since $x\alpha_1 \in dom(\alpha_2)$) = $(x\alpha_1^*)\alpha_2^* = x\alpha_1^*\alpha_2^*$ and so $\alpha \to \alpha^*$ is a homomorphism. Suppose $\alpha_1^* = \alpha_2^*$. Then $x\alpha_1^* = x\alpha_2^* \Rightarrow x\alpha_1 = x\alpha_2 \Rightarrow \alpha_1 = \alpha_2$. Also if $\alpha^* \in P_n^*$; then by the very definition of α^*, there exists $\alpha \in P_n$ such that $\alpha \to \alpha^*$. Hence $P_n \to P_n^*$ is an isomorphism. The semigroup U_n is in effect T_{n+1} and so the cardinal number $|U_n| = (n+1)^{n+1}$, and since $P_n^* = \{\alpha \in U_n: 0\alpha = 0\}$, we have $|P_n^*| = (n+1)^n$ and so $|P_n| = (n+1)^n$. The diameter of a graph Γ is the largest distance in Γ, that is $d(\Gamma) = \max\{d(v,w): v, w \in V(\Gamma)\}$. An element $\alpha \in T_n$ is called *quasi-nilpotent* if and only if α^k is a contant mapping for some positive integer k. An element $\alpha \in Sing_n$ is quasi-nilpotent if and only if $fix(\alpha) = 1$ and there is no non-empty subset A of X_n (except $F(\alpha)$) such that $A\alpha = A$. Every idempotent $\alpha \in Sing_n$ is expressible as a product of two quasi-nilpotents. Every element $\alpha \in Sing_n$ of height $r \le n-1$ is expressible as a product of three quasi-nilpotents of the same height. The index of quasi-nilpotency of α is the diameter of the digraph representing α.

Imam *et al.* (2024) studied quasi-idempotent elements in the semigroup of partial order-preserving transformations and showed that semigroup $\mathcal{PO}_n$ is quasi-idempotent generated, and that the upper bound for quasi-idempotent rank of $\mathcal{PO}_n$ is $\left\lceil\frac{5n-4}{2}\right\rceil$ (where $\lceil x\rceil$ denotes the least positive integer m such that $x \le m \le x+1$). That for any $\alpha \in \mathcal{PO}_n$ (the semigroup of partial order-preserving transformations), if $\alpha = \alpha^2$ then α is called an idempotent; and if $\alpha \ne \alpha^2 = \alpha^4$ then α is called a quasi-idempotent.

Given that $QE_1 = \{\beta_1, \beta_2, \dots, \beta_{n-1}, \alpha_1, \alpha_2, \dots, \alpha_{n-1}, \mu_2, \mu_3, \dots, \mu_{n-2}, \delta_2, \delta_3, \dots, \delta_{n-2}\}$, where $\mu_i = \begin{pmatrix} i & i+1 \\ i-1 & i \end{pmatrix}$ for $i = 2, \dots, n-1$, is decreasing and $\delta_i = \begin{pmatrix} i-1 & i \\ i & i+1 \end{pmatrix}$ is increasing quasi-idempotents. Also, $\beta_i = \begin{pmatrix} i+1 \\ i \end{pmatrix}$ is a decreasing quasi-idempotents and $\alpha_i = \begin{pmatrix} i-1 \\ i \end{pmatrix}$ is increasing. Thus, $\alpha = \beta_i\beta_{i+1}\dots\beta_{j-1} = \begin{pmatrix} i+1 \\ i \end{pmatrix}\begin{pmatrix} i+2 \\ i+1 \end{pmatrix}\dots\begin{pmatrix} j \\ j-1 \end{pmatrix}$ and $\alpha = \alpha_{i-1}\alpha_{i-2}\dots\alpha_{j+1} = \begin{pmatrix} i-1 \\ i \end{pmatrix}\begin{pmatrix} i-2 \\ i-1 \end{pmatrix}\dots\begin{pmatrix} j \\ j+1 \end{pmatrix}$.

MATERIALS AND METHODS

Let $d(\alpha^*) = |X_n \backslash im(\alpha^*)|$ as defect of $\alpha^* \in P_n$, and $[x_1, x_2, \dots, x_m|x_m]$ be an m-path (Ayik *et al.*, 2005) and $[x_1, x_2, \dots, x_m|0]$ be an m-chain. Then $\beta^* \in P_n$ is said to be a 3-chain if $\beta^* = [x_1, x_2, x_3|0]$. If $x_3 = 0$, we get a 2-chain $[x_1, x_2|0]$ and if $x_2 = x_3 = 0$, then $\gamma \in P_n$ becomes a one-chain $\gamma = [x_i|0] = \begin{pmatrix} x_1 & x_2 \dots & x_{i-1}x_{i+1}\dots x_n \\ x_1 & x_2 \dots & x_{i-1}x_{i+1}\dots x_n \end{pmatrix}$. In this regard, $d(\gamma) = 0$, ie γ has defect as 0; and $\alpha \in P_n$ is a product of n one-chains as $[x_1|0][x_2|0][x_3|0][x_4|0]\dots[x_n|0]$. Since $\gamma^2 = [x_1|0][x_1|0] = [x_1|0] = \gamma$, then γ is an idempotent having rank n if and only if γ has defect 0. Since $[x_1|0][x_2|0][x_3|0][x_4|0]\dots[x_n|0] = \begin{pmatrix} x_1x_2x_3\dots x_n \\ \emptyset \;\; \emptyset \;\;\; \emptyset \dots \emptyset \end{pmatrix}$, then $idrank(\gamma) = rank(\gamma)$ and the nilpotent γ is idempotent generated as customary to the existing facts (Ayik *et al.*, 2005). When $n - |im(\alpha)| = 1$, then $\alpha^* \in P_n$ has the length of the products of its nilpotents as $\frac{n(n-1)}{2}$ (Imam and Ibrahim, 2022), analogous to the idempotent rank of $\alpha \in T_n$ whence $\alpha^* = [x_1, x_2|0][x_3, x_4|0][x_5, x_6|0]\dots[x_{n-1}, x_n|0]$ (even if α is not ordered). The $\frac{n(n-1)}{2}$ is the choices of 2 out of n-number of elements in X_n. This corresponds to the Stirling number of the second kind $S(n, n-1) = \frac{n(n-1)}{2}$ which also befits the Vagner Theorem. Analogous to the fact that not both $[x_1, x_2|x_2]$ and $[x_2, x_1|x_1]$ are assemble in the products, then the $nilrank_{2,0}(P_n) = \frac{n(n+1)}{2}$.

Let $g(\alpha^*)$ denotes the gravity of $\alpha^* \in P_n$, then $f(\alpha^*) = |Fix(\alpha^*)|$ is the cardinality of the set of acyclic, trivial and terminal orbits. For a 3-chain having a cycle of size r, then $g(\alpha^*)$ is maximized by increasing the number of 2-cycles and decreasing the number of fixed points. When r is even, we have $\frac{r}{2}$ two-cycles and $\frac{1}{2}(r-1)$ two-cycles when r is odd. Thus, $g(\alpha^*) = n^* + c(\alpha^*) - f(\alpha^*) = (n+1) + c(\alpha^*) - f(\alpha^*) = n + \frac{1}{2}(r-1) - 2$. When $r = n+1$, we have $g(\alpha^*) = (n+1) + \frac{1}{2}(n) - 2 = \frac{2(n+1)+n-4}{2} = \frac{3n-2}{2}$. When n is odd, $g(\alpha^*) = (n+1) + \frac{1}{2}n - 2 = \frac{3n-2}{2}$.

Let $X_n^0 = X_n \cup \{0\}$ and let P_n be the partial transformation semigroup on X_n^0. If $\{x_0, x_1, \dots, x_m\} \subseteq X_n^0$ (where $x_0 = 0$) and $\alpha^* \in P_n$ is define by $x_i\alpha^* = x_{i+1}, x_m\alpha^* = x_r (1 \le r \le m)$ and $x\alpha^* = x (x \in X_n \backslash \{x_1, x_2, \dots, x_m\})$, then α^* is called a path-cycle of length m and period r, or simply, an (m,r)-path-cycle, and is denoted (in a linear notation) by $\alpha^* = [x_1, \dots, x_m|x_r]$. If $r = m$, α^* is called an m-path to x_m or simply an m-path; if $m \ge 2$ and $r = 1$, α^* is called an m-cycle; if $m = r = 1$, α^* is called a loop; if $m = r = 0$, α^* is called a terminal; if $m = r = 2$, α^* is an idempotent of defect one; if $m \ge 2$ and $r \ne 0$ or 1,

α^* is said to be a proper path-cycle; if $r = 0$, α^* is an m-chain or a nilpotent of index m.

Let $\xi^* = [x_1, x_2, x_3|x_0]$ be an arbitrary 3-chain of α^*, ξ^* maps x_1 to x_2, x_2 to x_3, x_3 to x_0, and all other elements of X_n^0 identically. Then ξ^* has a *linear notation* as $\xi^* = [x_1, x_2, x_3|0] = \begin{pmatrix} x_1 & x_2 x_3 \\ x_2 & x_3\ \emptyset \end{pmatrix}$ because of the Vagner Representation Theorem. We shall refer to x_1 as the *first entry*, x_2 as the *middle entry* (or *second entry*), x_3 as the *third entry* and x_0 as *no entry* of ξ^*.

Let $\alpha^* \in P_n$. The equivalence relation $\sim = \{(x, y) \in X_n^0 \times X_n^0 : (\exists u, v \geq 0) x\alpha^{*u} = y\alpha^{*v}\}$, partitioned X_n^0 into orbits $\Omega_1, \Omega_2, \ldots, \Omega_k$. These orbits correspond to the connected components of the digraph associated to α^* with vertex set X_n^0 in which there is a directed edge (x, y) if and only if $x\alpha^* = y$. Each orbit Ω has a kernel defined by $K(\Omega) = \{x \in \Omega: (\exists r > 0) x\alpha^{*r} = x\}$.

Every orbit of α^* falls into exactly one of these five categories and all the five cases can arise for a single $\xi^* \in P_n$. Let $c(\alpha^*)$ be the number of cyclic orbits of α^* and $f(\alpha^*)$ be the number of fixed points of α^* which is equal to the sum of the number of terminal, trivial and acyclic orbits of α^*. The gravity of α^* is define as $g(\alpha^*) = n^* + c(\alpha^*) - f(\alpha^*)$, where $n^* = n + 1$.

For each standard or acyclic orbit Ω of $\alpha^* \in P_n$ and each $x \in \Omega \backslash \mathrm{im}(\alpha)$, the sequence $x, x\alpha^*, x\alpha^{*2}, x\alpha^{*3}, \ldots$ eventually arrives in $K(\Omega)$, and remains there for all subsequent iterations. Denote the set of all distinct elements in this sequence by $Z(x)$. Suppose that $\alpha^* \in P_n \backslash S_n$ has s standard orbits $\Omega_1, \Omega_2, \ldots, \Omega_s$. For each $j = 1,2, \ldots, s$, let $\Omega_j \backslash im(\alpha^*) = \{x_{1j}, x_{2j}, \ldots, x_{k_j j}\}$, where x_{1j} is such that
$$|Z(x_{1j})| = \begin{cases} max_{1 \leq i \leq k_j}\{|Z(x_{ij})| : |Z(x_{ij})| \text{is even}\} \\ max_{1 \leq i \leq k_j}\{|Z(x_{ij})| : |Z(x_{ij})| \text{is odd}\} \end{cases}.$$
Then there exist $m_j \geq 1$ and $r_j \geq 2$ such that $K(\Omega_j) = \{x_{1j}\alpha^{*m_j}, x_{1j}\alpha^{*m_j+1}, \ldots, x_{1j}\alpha^{*m_j+r_j-1}\}$, where $x_{1j}\alpha^{*m_j+r_j} = x_{1j}\alpha^{*m_j}$. Note that this definition of $K(\Omega_j)$ is still valid for every x_{ij}, not only for x_{1j}, and moreover, they are all the same.

Let
$$Z_1(\Omega_j) = Z(x_{1j}) = \{x_{1j}, x_{1j}\alpha^*, \ldots, x_{1j}\alpha^{*m_j}, x_{1j}\alpha^{*m_j+1}, \ldots, x_{1j}\alpha^{*m_j+r_j-1}\}$$
and $Z_i(\Omega_j) = \{x_{ij}, x_{ij}\alpha, \ldots, x_{ij}\alpha^{p_{ij}-1}\}$ $(2 \leq i \leq k_j)$ where $x_{ij}\alpha^{p_{ij}} \in (Z_1(\Omega_j) \cup Z_2(\Omega_j) \cup Z_3(\Omega_j) \cup \ldots \cup Z_{i-1}(\Omega_j))$. Then $\{Z_i(\Omega_j): 1 \leq i \leq k_j\}$ is a partition of Ω_j.

Suppose that $\alpha^* \in P_n \backslash S_n$ has acyclic or terminal orbit; let Φ be the union of all its acyclic orbits and let Ξ be the union of all its terminal orbits, and denote the set $\{x \in \Phi: x\alpha^* = x\}$by $Fix(\Phi)$. Let $\Phi \backslash im(\alpha^*) = \{x_1, x_2, \ldots, x_l\}$ where x_1 is such that $|Z(x_1)| = \begin{cases} max_{1 \leq i \leq k_j}\{|Z(x_u)| : |Z(x_u)| \text{is even}\} \\ max_{1 \leq i \leq k_j}\{|Z(x_u)| : |Z(x_{ij})| \text{is odd}\} \end{cases}$. Then, for $u = 1,2, \ldots, l$ define $Y_u(\Phi) = \{x_u, x_u\alpha^*, \ldots, x_u\alpha^{*q_u-1}\}$, where $x_1\alpha^{*q_1} \in Fix(\Phi)$ and $x_u\alpha^{*q_u} \in (Y_1(\Phi) \cup Y_2(\Phi) \cup \ldots \cup Y_{u-1}(\Phi) \cup Fix(\Phi))(u = 2,3, \ldots, l)$. Thus, $\{Y_u(\Phi): 1 \leq u \leq l\}$ is a partition of Φ. We will be interested in the cardinalities of $Z_i(\Omega_j)$ and $Y_u(\Phi)$ being odd or even. For this, we define indicator functions z_{ij} and $y_u$by $z_{ij} = \begin{cases} 0 \text{ if} |Z_i(\Omega_j)| \text{is even} \\ 1 \text{ if} |Z_i(\Omega_j)| \text{is odd} \end{cases}$,
$y_u = \begin{cases} 0 \text{ if } |Y_u(\Phi)| \text{is even} \\ 1 \text{ if } |Y_u(\Phi)| \text{is odd} \end{cases}$ and
$w_u = \begin{cases} 0 \text{ if } |W_u(\Xi)| \text{is even} \\ 1 \text{ if } |W_u(\Xi)| \text{is odd} \end{cases}$. For each $\alpha^* \in P_n \backslash S_n$, we define the measure of α^* by $m(\alpha^*) = \begin{cases} l(\alpha^*) - e(\alpha^*) \text{if } l(\alpha^*) > e(\alpha^*) \\ 0 \text{ if } l(\alpha^*) \leq e(\alpha^*) \end{cases}$, where $l(\alpha^*) = \sum_{j=1}^{s} \sum_{i=1}^{k_j} z_{ij} + \sum_{u=1}^{l} y_u + \sum_{v=1}^{b} w_u$ and $e(\alpha^*)$ denote the number of cyclic orbits of α^* of even cardinality.

RESULTS AND DISCUSSION

An element $\alpha^* \in P_n: X_n \cup \{0\} \to X_n \cup \{0\}$ is said to be a nilpotent element if there exists a positive integer m such that $\alpha^{*m} = 0$. Whenever $\alpha^{*i} = 0 \Longrightarrow x\alpha^{*i} = 0 \Longrightarrow x\alpha^{*i} = 0 = y\alpha^{*j}$ which is the terminal orbit generated by the equivalence relationship $x \equiv y$ whenever $x\alpha^{*i} = y\alpha^{*j}$. Thus P_n has one orbit ahead of T_n, the finite set of full transformations on $X_n = \{1,2, \ldots, n\}$. Since every semigroup is embeddable in regular idempotent generated semigroups (Howie, 1966) such as T_n, then $|P_n| = (n + 1)^n$ is embeddable in $T_{n+1} = (n + 1)^{(n+1)}$ defined on $X_n \cup \{0\}$. This is because there is a bijective morphism in the map $\vartheta: P_n \to T_{n+1}$ defined by $\vartheta(\alpha^*|_{X_n}) = \alpha|_{X_n \cup \{0\}}$ since $\vartheta(\alpha^*|_{X_n} \circ \beta^*|_{X_n}) = \vartheta((\alpha^* \circ \beta^*)|_{X_n})$ which by definition is equal to $(\alpha \circ \beta)|_{X_n \cup \{0\}} = \alpha|_{X_n \cup \{0\}} \circ \alpha|_{X_n \cup \{0\}} = \vartheta(\alpha^*|_{X_n}) \circ \vartheta(\beta^*|_{X_n})$. The oneness and ontoness is by the fact that $\vartheta(\alpha^*|_{X_n}) = \vartheta(\beta^*|_{X_n}) \Leftrightarrow \alpha|_{X_n \cup \{0\}} = \alpha|_{X_n \cup \{0\}} \Leftrightarrow \alpha^*|_{X_n} = \beta^*|_{X_n}$. The map $\vartheta: P_n \to T_{n+1}$ is categorically $\vartheta(\alpha^*|_{X_n \cup \emptyset}) = \alpha|_{X_n \cup \{0\}}$, but merely represented by $\vartheta(\alpha^*|_{X_n}) = \alpha|_{X_n \cup \{0\}}$ since empty set is a subset of every set. A map $\vartheta: P_n \to P_n^* \subseteq T_n$ is the Vagner Theorem usable in translating results of T_n to P_n through P_n^* (Saito, 1989).

Example 1: Consider the map

$$\alpha^* = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 \\ 2 & 3 & 4 & 2 & 4 & 7 & 8 & 7 & 7 & 11 & 11 & 13 & 14 & \emptyset & 16 & 17 & 15 & 19 & 18 & 21 & 20 & 22 \end{pmatrix} \in P_n$$ with orbits

$standard$: $\Omega_1 = \{1,2,3,4,5\}, \Omega_2 = \{6,7,8,9\}$

$acyclic$: $\Phi_1 = \{10,11\}$

$cyclic$: $\Theta_1 = \{15,16,17\}, \Theta_2 = \{18,19\}, \Theta_3 = \{20,21\}$

$trivial$: $\Psi_1 = \{22\}$

$terminal$: $\Xi_1 = \{12,13,14\}$

As shown in Figure 1 below:

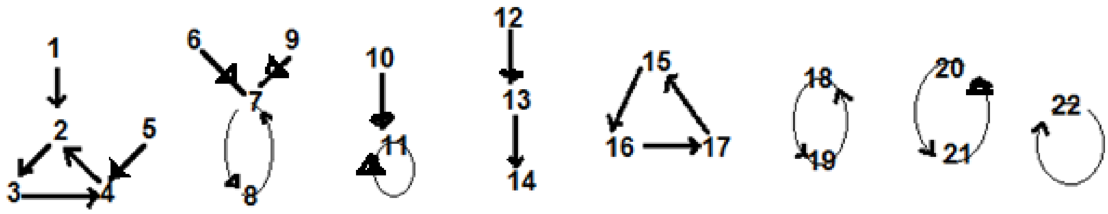

Figure 1: Orbits of $\alpha^* \in P_{22}$

For this α^*, we have $\Phi = \{10,11\}$ and so $Z_1(\Omega_1) = \{2,3,4,5\}$ and $Z_2(\Omega_1) = \{1\}, Z_1(\Omega_2) = \{6,7,8\}, Z_2(\Omega_2) = \{9\}, Y_1(\Phi) = \{10\}, W_1(\Xi) = \{12,13,14\}$. Thus, $z_{11} = 0, z_{21} = 1, z_{12} = 1, z_{22} = 1, y_1 = 0, w_1 = 0$, and so, $l(\alpha^*) = z_{11} + z_{21} + z_{21} + z_{22} + y_1 + w_1 = 4$, also $e(\alpha) = 2$. Therefore the measure of $\alpha^*$is $m(\alpha^*) = 2$.

Theorem 3.1. Let E be the set of all idempotents in P_n and E_1 be the set of all idempotents of defect 1 in E. For $n \geq 3$, each $\alpha^* \in E_1$ is expressible as a product of nilpotents in P_n.

Proof. Each idempotent in E_1 is of the form $[i,j|j]$, with $i,j \in X_n$ and $i \neq j$. Thus, since $n \geq 3$, there is a $k = 0 \in X_n^0$ such that $[i,j|j] = [i,j,k|0][i,j,k|0]$.

Theorem 3.2. For $n \geq 3$, each $\alpha \in E \backslash E_1$ is expressible as a product of $g(\alpha^*)$ nilpotents of defect 1 in P_n.

Proof. Let $\alpha^* \in E \backslash E_1$ and let $A_1, A_2, \dots, A_r$ be its non-singleton blocks. Then, each of the blocks $A_i (1 \leq i \leq r)$ is stationary. If $|A_i| \geq 3$ for some i, we assume without loss of generality that $|A_1| \geq 3$. Let $A_i \backslash \{A_i\alpha^*\} = \{x_{i1}, x_{i2}, \dots, x_{ia_i}\} (1 \leq i \leq r)$ and define products $\xi_i (1 \leq i \leq r)$ of nilpotents of 3-chains by

$\xi_1 = [x_{11}, x_{12}, A_1\alpha^*|0][x_{11}, x_{13}, A_1\alpha^*|0][x_{11}, x_{14},, A_1\alpha^*|0] \dots [x_{11}, x_{1a_1}, A_1\alpha^*|0][x_{11}, x_{12}, A_1\alpha^*|0]$ and $\xi_i = [x_{11}, x_{i2}, A_i\alpha^*|0][x_{11}, x_{i3}, A_i\alpha^*|0][x_{11}, x_{14}, A_i\alpha^*|0] \dots [x_{i1}, x_{ia_i}, A_i\alpha^*|0] (2 \leq i \leq r)$. Then it is easy to verify that $\alpha^* = \xi_1\xi_2 \dots \xi_r$. Also, each point in $A_i \backslash \{A_i\alpha^*\} (2 \leq i \leq r)$ appeared exactly once as second entry of a 3-chain in ξ_i and each point in $A_1 \backslash \{A_1\alpha^*, x_{11}, x_{12}\}$ appeared exactly once as a second entry of a 2-chain in ξ_1. The point x_{12} appeared exactly once as a second entry of the 3-chains in ξ_1 while the point x_{11} did not appear anywhere as a second entry. Thus, the number of 3-chains used in the product $\xi_1\xi_2 \dots \xi_r$is $\sum_{i=1}^{r} |A_i \backslash \{A_i\alpha^*\}| = n - f(\alpha^*) = g(\alpha^*)$. If $|A_i| = 2$ for all i, let $A_i = \{x_i, x_i\alpha^*\} (1 \leq i \leq r)$. Then $\alpha^* = [x_r, x_1, x_1\alpha|0][x_r, x_2, x_2\alpha|0] \dots [x_r, x_{r-1}, x_{r-1}\alpha|0][x_r, x_1, x_r\alpha|0]$ and again, the number of 3-chains used is $n - f(\alpha^*) = g(\alpha^*)$.

Example 2: Consider the element

$$e = \begin{pmatrix} \{1,2,7,5\} & \{3,8,10,12\} & \{4,6,9,11\} & \{13,14,15,16\} \\ 2 & 8 & 11 & 0 \end{pmatrix} = \xi_1\xi_2\xi_3\xi_0$$ where

$$\xi_1 = [1,5,2|2][1,7,2|2][1,5,2|2]$$

$$\xi_2 = [1,3,8|8][1,10,8|8][1,12,8|8]$$

$$\xi_3 = [1,4,11|11][1,6,11|11][1,9,11|11]$$

$$\xi_0 = [1,13,14|0][1,14,15|0][1,15,16|0]$$

Theorem 3.3. For $n \geq 3$, each $\alpha^* \in P_n \backslash E$ is expressible as a product of $\left\lceil \frac{1}{2}(g(\alpha) + m(\alpha)) \right\rceil$ 3-chain nilpotents in $P_n \backslash S_n$.

Proof. Suppose that $\alpha^* \in P_n \backslash E$ has orbits as follows, $standard$: $\Omega_1, \Omega_2, \dots, \Omega_s$, $acyclic$: $\Phi_1, \Phi_2, \dots, \Phi_a$, $cyclic$: $\Theta_1, \Theta_2, \dots, \Theta_c$, $trivial$: $\Psi_1, \Psi_2, \dots, \Psi_t$, $terminal$: $\Xi_1, \Xi_2, \dots, \Xi_b$. For each standard orbit Ω_j, let $\Omega_j \backslash im(\alpha) = \{x_{1j}, x_{2j}, \dots, x_{k_j j}\}$; $K(\Omega_j) = \{x_{1j}\alpha^{*m_j}, x_{1j}\alpha^{*m_j+1}, \dots, x_{1j}\alpha^{*m_j+r_j-1}\}$, let $Z_1(\Omega_j) = Z(x_{1j})$
$= \{x_{1j}, x_{1j}\alpha^*, \dots, x_{1j}\alpha^{*m_j}, x_{1j}\alpha^{*m_j+2}, \dots, x_{1j}\alpha^{*m_j+r_j-1}\}$ and let $Z_i(\Omega_j) = \{x_{ij}, x_{ij}\alpha, \dots, x_{ij}\alpha^{p_{ij}-1}\}$ $(2 \leq i \leq k_j)$, where $x_{ij}\alpha^{p_{ij}} \in (Z_1(\Omega_j) \cup Z_2(\Omega_j) \cup Z_3(\Omega_j) \cup \dots \cup Z_{i-1}(\Omega_j))$. Let $\Phi = \Phi_1 \cup \Phi_2 \cup \dots \cup \Phi_a$; $\Phi \backslash \text{im}(\alpha) = \{x_1, x_2, \dots, x_l\}$ and define $Y_u(\Phi) = \{x_u, x_u\alpha^*, \dots, x_u\alpha^{*q_u-1}\}$, where $x_1\alpha^{*q_1} \in Fix(\Phi)$ and $x_u\alpha^{*q_u} \in (Y_1(\Phi) \cup Y_2(\Phi) \cup \dots \cup Y_{u-1}(\Phi) \cup Fix(\Phi))(u = 2,3, \dots, l)$. Let $\Theta_v = \{y_v, y_v\alpha, \dots, y_v\alpha^{p_v-1}\}$ (where $y_v\alpha^{p_v} = y_v$). Then we consider six possible cases that may arise.

Case 1 ($0 = e(\alpha) = l(\alpha)$). In this case each $Z_i(\Omega_j)(i = 1,2, \dots, k_j)$ and each $Y_u(\Phi)(u = 1,2, \dots, l)$ is of even size; also, each Θ_v is of odd size. Thus, corresponding to each $Z_1(\Omega_j), Z_i(\Omega_j)(i = 2,3, \dots, k_j)$, $Y_u(\Phi)(u = 1,2, \dots, l)$, $\Theta_v(v = 1,2, \dots, c)$ and $\Xi_w(w = 1,2, \dots, a)$; we define, respectively, products $\xi_{1j}, \xi_{ij}(i = 2,3, \dots, k_j)$, $\tau_u(u = 1,2, \dots, l)$, $\eta_v(v = 1,2, \dots, c)$ and $\phi_w(w = 1,2, \dots, b)$ of 3-chains by

$$\xi_{1j} = [x_{1j}, x_{1j}\alpha^*, x_{1j}\alpha^{*2}|0][x_{1j}\alpha^{*3}, x_{1j}\alpha^{*4}, x_{1j}\alpha^{*5}|0] \dots [x_{1j}\alpha^{*m_j+r_j-2}, x_{1j}\alpha^{*m_j+r_j-1}, x_{1j}\alpha^{*m_j-1}|0]$$
$$\xi_{ij} = [x_{ij}, x_{ij}\alpha^*, x_{ij}\alpha^{*2}|0][x_{ij}\alpha^{*3}, x_{ij}\alpha^{*4}, x_{ij}\alpha^{*5}|0] \dots [x_{ij}\alpha^{*m_j+r_j-2}, x_{ij}\alpha^{*m_j+r_j-1}, x_{ij}\alpha^{*m_j-1}|0]$$
$$\tau_u = [x_u, x_u\alpha^*, x_u\alpha^{*2}|0][x_u\alpha^{*3}, x_u\alpha^{*4}, x_u\alpha^{*5}|0] \dots [x_u\alpha^{*q_u-2}, x_u\alpha^{*q_u-1}, x_u\alpha^{*q_u}|0]$$
$$\eta_v = [y_v, y_v\alpha^*, y_v\alpha^{*2}|0][y_v\alpha^{*3}, y_v\alpha^{*4}, y_v\alpha^{*5}|0] \dots [y_v\alpha^{*q_u-2}, y_v\alpha^{*q_v-1}, y_v\alpha^{*q_v} = y_v|0]$$
$$\phi_w = [z_w, z_w\alpha^*, z_w\alpha^{*2}|0][z_w\alpha^{*3}, z_w\alpha^{*4}, z_w\alpha^{*5}|0] \dots [z_w\alpha^{*q_w-2}, z_w\alpha^{*q_w-1}, y_w\alpha^{*q_w}|0]$$

For each $j = 1,2, \dots, s$, let $\beta_j = \xi_{1j}\xi_{2j} \dots \xi_{k_j j}$, then each element $x \in \Omega_j$ appears exactly once either as a first entry or a second entry of a 3-chain in the product β_j. Moreover, with the sole exception of $x = x_{ij}\alpha^{*m_j-1}$, an element $x \in \Omega_j$ appearing as the third entry never subsequently reappears as an upper or middle entry. Hence each $x \neq x_{ij}\alpha^{*m_j+r_j-1}$ in Ω_j is moved by exactly one of the 3-chains appearing in the product β_j and moreover, it is moved to $x\alpha$. The exceptional element $x_{ij}\alpha^{*m_j+r_j-1}$ is moved to $x_{ij}\alpha^{*m_j-1}$ by the first 3-chain in the product ξ_{1j} and then is moved, by either $[x_{1j}\alpha^{*m_j-2}, x_{1j}\alpha^{*m_j-1}, x_{1j}\alpha^{*m_j}]$ or $[x_{1j}\alpha^{*m_j-1}, x_{1j}\alpha^{*m_j}, x_{1j}\alpha^{*m_j+1}]$ to $x_{1j}\alpha^{*m_j} = x_{1j}\alpha^{*m_j+r_j}$. Thus, $x\beta_j = x\alpha$ for every $x \in \Omega_j$, while $x\beta_j = x$ for every $x \in X_n \backslash \Omega_j$. Since the orbits $\Omega_j (1 \leq j \leq s)$ are pairwise disjoint, we have a product of 3-chains such that $x\beta_1\beta_2 \dots \beta_s = \begin{cases} x\alpha & \text{if } x \in \bigcup_{j=1}^{s} \Omega_j \\ x & \text{if } x \in X_n \backslash \bigcup_{j=1}^{s} \Omega_j \end{cases}$.

Similarly, if $\gamma = \tau_1\tau_2 \dots \tau_l$ then each point $x \in \Phi$ appears either as a first entry or second entry of a 3-chain in the product γ. Moreover, each $x \in \Phi$ that appears as a third entry or second entry never subsequently reappears as a first or middle entry. Hence each $x \in \Phi$ is moved to $x\alpha^*$ by exactly one of the 3-chains appearing in the product γ. Thus, $x\gamma = x\alpha$ for each $x \in \Phi$ while $x\gamma = x$ for each $\in X_n \backslash \Phi$.

Also, if $\delta = \eta_1\eta_2 \dots \eta_c$, then, again, we can observe that the product δ is such that $x\delta = x\alpha^*$ for each $x \in \bigcup_{v=1}^{c} \Theta_v$ and $x\delta = x$ for each $x \in X_n \backslash \bigcup_{v=1}^{c} \Theta_v$. Hence, it follows that $\alpha = \beta_1\beta_2 \dots \beta_s\gamma\delta$ a product of 3-chains in $P_n \backslash S_n$. Let us denote the number of 3-chains in the product ξ_{ij}, τ_u and η_v by $\#(\xi_{ij}), \#(\tau_u)$ and $\#(\eta_v)$ respectively (we shall also use similar notation in the sequel).

Then, counting the number of points appearing at the first position of each product ξ_{ij}, τ_u and η_v, we have $\#(\xi_{ij}) = \frac{1}{2}|Z_i(\Omega_j)|$(even), $\#(\tau_u) = \frac{1}{2}|Y_u(\Phi)|$ (even) and $\#(\eta_v) = \frac{1}{2}(|\Theta_v| + 1)$. And so, $\#(\beta_j) = \frac{1}{2}\sum_{i=1}^{k_j}|Z_i(\Omega_j)| = \frac{1}{2}|\Omega_j|$, so that $\#(\beta_1\beta_2 \dots \beta_s) = \frac{1}{2}\sum_{j=1}^{s}|\Omega_j|$, $\#(\gamma) = \frac{1}{2}\sum_{u=1}^{l}|Y_u(\Phi)|$ and $\#(\delta) = \frac{1}{2}\sum_{v=1}^{c}(|\Theta_v| + 1) = \frac{1}{2}(\sum_{v=1}^{c}|\Theta_v| + c)$. Using these, while noting that $\sum_{j=1}^{s}|\Omega_j| + \sum_{v=1}^{c}|\Theta_v| + \sum_{u=1}^{l}|Y_u(\Phi)| = n - (a + t)$, we have $\#(\alpha^*) = \frac{1}{2}\left(n + c - (a + t)\right) = \frac{1}{2}(n +$ $c\alpha* - f\alpha* = g(\alpha*)2$.

The other cases $0 = l(\alpha^*) < e(\alpha^*)$, $0 = e(\alpha^*) < l(\alpha^*)$, $0 < l(\alpha^*) = e(\alpha^*)$, $0 < l(\alpha^*) < e(\alpha^*)$, $0 < e(\alpha) < l(\alpha)$ followed analogously to the cases presented in Imam and Ibrahim (2022). However, the case $0 < l(\alpha^*) < e(\alpha^*)$ yields a complete package as $\#(\alpha^*) = \frac{1}{2}\left(n - (a + t) + \sum_{j=1}^{s}\sum_{i=1}^{k_j} z_{ij} + \sum_{u=1}^{l} y_u + \right.$ $c - e = 12n + c\alpha* - f\alpha* + l\alpha* - e\alpha* = 12(g\alpha* + m(\alpha*)).$

Hence in all cases, the length of the products of nilpotents is $\left\lceil \frac{1}{2}(g(\alpha^*) + m(\alpha^*)) \right\rceil$.

Example 3: Let $\alpha^* = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 \\ 2 & 3 & 4 & 2 & 4 & 7 & 8 & 7 & 7 & 11 & 11 & 13 & 14 & \emptyset & 16 & 17 & 15 & 19 & 18 & 21 & 20 & 22 \end{pmatrix} \in P_n$ with orbits $standard$: $\Omega_1 = \{1,2,3,4,5\}, \Omega_2 = \{6,7,8,9\}$, $acyclic$: $\Phi_1 = \{10,11\}$, $cyclic$: $\Theta_1 = \{15,16,17\}, \Theta_2 = \{18,19\}, \Theta_3 = \{20,21\}$, $trivial$: $\Psi_1 = \{22\}$, $terminal$: $\Xi_1 = \{12,13,14\}$ we have $\Phi = \{10,11\}$ and so $Z_1(\Omega_1) = \{2,3,4,5\}$ and $Z_2(\Omega_1) = \{1\}, Z_1(\Omega_2) = \{6,7,8\}, Z_2(\Omega_2) = \{9\}, Y_1(\Phi) = \{10\}, W_1(\Xi) = \{12,13,14\}$. Thus, $z_{11} = 0, z_{21} = 1, z_{12} = 1, z_{22} = 1, y_1 = 0, w_1 = 0$, and so, $l(\alpha^*) = z_{11} + z_{21} + z_{21} + z_{22} + y_1 + w_1 = 4$, also $e(\alpha^*) = 2$. Therefore, the measure of $\alpha^*$is $m(\alpha^*) = 2$, so that α^* is expressed as a product of $k(\alpha^*) = \left\lceil \frac{1}{2}(22+2) \right\rceil = 12$ three-chains in P_{22}. This gives $\alpha = \xi_{11}\xi_{12}\xi_{21}\xi_{22}\tau_1\tau_2\eta_1\eta_2\eta_3$, where $\xi_{11} = [1,2,3|0][3,\ 4,\ 1|0], \xi_{12} = [18,\ 5,\ 4|0], \xi_{21} = [1,\ 6,\ 7|0][7,\ 8,\ 6|0], \xi_{22} = [20,\ 9,\ 7|0], \tau_1 = [12,\ 13,\ 14|0], \tau_2 = [12,\ 10,\ 11|0], \eta_1 = [1,\ 16,\ 17|0][17,\ 15,\ 1|0], \eta_2 = [5,\ 19,\ 18|0], \eta_3 = [9,\ 21,\ 20|0]$.

CONCLUSION

Let $\alpha \in Sing_n = T_n \backslash S_n = K(n, n-1)$, where $K(n,r) = \{\alpha \in T_n : |im\ \alpha| \le r\}$. Then the idempotent depth of products of idempotents of defect 1 is $\frac{n(n-1)}{2} = S(n, n-1)$, where $S(n,r)$ is the Stirling number of the second kind and T_n is the semigroup of full transformations on $X_n = \{1,2,\dots,n\}$. Since the set of idempotents of defect d and of rank $n-d$ written as E_d are generate-able by the idempotents of defect 1 given by $\varepsilon_1 = \binom{i}{j}$, and $\alpha = \varepsilon_1\varepsilon_2 \dots \varepsilon_{k(\alpha)}$, where $k(\alpha) = \left\lceil \frac{g(\alpha)}{d(\alpha)} \right\rceil$ as given by Saito (1989) and $k(\alpha) = \frac{1}{2}[g(\alpha) + m(\alpha)]$ accorded by Imam and Ibrahim (2022) who considered 3-path analogue of idempotents of defect 2; this work amalgamated the two depth expressions into $k(\alpha) = \left\lceil \frac{g(\alpha)+m(\alpha)}{d(\alpha)} \right\rceil$; where $k(\alpha)$ is the *depth* of α, $m(\alpha)$ is the *measure* of α and $d(\alpha)$ is the *defect* of α expressible as $d(\alpha) = n - r(\alpha)$ and $r(\alpha)$ is the *rank* of α.

Thus, the idempotents of rank 1 have defect $n-1$. If $m(\alpha) = 0$or $m(\alpha) = d(\alpha)$, where $\frac{n}{r(\alpha)}$ is a positive integer, then we retrieve back the Saito's result. When $d(\alpha) = 2$, we get back the case study of Imam and Ibrahim (2022). Since $\lceil x \rceil$ is the greatest integer m less than or equal to x and $m(\alpha)$ is a variable, then $k(\alpha) = \left\lceil \frac{g(\alpha)+m(\alpha)}{d(\alpha)} \right\rceil + 1$ as expected has been handled by the variability of $m(\alpha)$; where, in effect, $0 \le m(\alpha) \le d(\alpha)(\forall \alpha \in Sing_n)$. For $\alpha \in P_n \backslash S_n$, the $\frac{n(n+1)}{2} = S(n+1, n)$ idempotents generate P_n.

Since $(12), \binom{1}{2}$ and $(1\ 2 \dots n)$ generate T_n, then $(12), \binom{1}{2}, (1\ 2 \dots n)$and $[1,2,\dots,n|0]$ generate P_n (for every n a positive integer and P_n the semigroup of partial maps, where 0 stands for empty map $\emptyset$ and $[1,2,\dots,n|0]$ is a digraphic chain). In fact, P_n is embeddable in the binary relation semigroup having the underlying set elements of the form $\binom{a}{b} = (a,b)$. Since P_n is a semigroup and every semigroup is embeddable in a regular idempotent generated rig-semigroup such as T_n, then P_n is embeddable in T_{n+1} which is apparent since $(n+1)^n = |P_n| \le (n+1)^{(n+1)} = |T_{n+1}|$. This however attested the Vagner Representation of P_n using $P_n^* \cong U_n \subseteq T_n$ once again, and $k(\alpha^*) = \left\lceil \frac{g(\alpha^*)+m(\alpha^*)}{d(\alpha^*)} \right\rceil$ whenever $\alpha^* \in P_n$.

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