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ON SIGNED FULL TRANSFORMATION SEMIGROUP OF A FINITE SET

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ABSTRACT
If we define $[n] = \{1,2,3,\dots,n\}$ and $[n^*] = \{\pm1,\pm2,\pm3,\dots,\pm n\}$, then the map $\alpha\colon [n] \to [n^*]$ is called a signed transformation on $[n]$. The collection of all these maps together with composition forms a semigroup called a signed transformation semigroup $\widetilde{T_n}$. Given that $dom(\alpha) = [n]$, $\widetilde{T_n}$ will be called a signed full transformation semigroup on $[n]$. In this paper, we obtained formulae that count the number of elements in the semigroups of order decreasing, order preserving and order decreasing signed transformations on $[n]$. We equally do same for the subsemigroup of $\widetilde{T_n}$ consisting only of idempotents.

Keywords: Semigroup, signed transformation, order preserving transformation, order decreasing transformation, idempotents

INTRODUCTION

Let $[n] = \{1,2,3,\dots,n\}$ and $[n^*] = \{\pm1,\pm2,\pm3,\dots,\pm n\}$ be finite sets. A map $\alpha\colon [n] \to [n]$ is called a transformation on [n]. If $[n] \to [n^*]$, α will be called a signed transformation on [n]. The collection of all signed transformations on [n] together with composition forms a semigroup called the signed transformation semigroup on the finite set [n]. We denote this semigroup by $\widetilde{T_n}$. $\widetilde{T_n}$ will be called a signed full transformation on $[n]$ since $dom(\alpha) = [n]$.

The concept of signed transformations was initiated by F. P. Richard in his work on transformation semigroups over groups in 2008. His study on the signed full transformation semigroup $\widetilde{T}_n$ as a semigroup analogue of the signed symmetric group $\widetilde{S}_n$ was an extension to the work of James and Kerber (1981), who studied the signed symmetric group. Since then, researchers have made tremendous headway in the study of signed transformations.They had studied this semigroups with respect to its algebraic and combinatorial properties. One of such works is the work by G. U. Garba in 1990 where he studied extensively on the idempotents in partial transformation semigroups. On the other hand, the collections of the signed transformations with the composition playing no role have also been objects of consideration by algebraists. Various semigroups of signed transformations have been studied in Mogbonju and Azeez, (2018). They considered signed order preserving transformation, partial order-preserving and order-preserving injective transformations. In all of these studies, their concern was on the combinatorial aspects of the semigroups. They deduced formulas that count the numbers of elements in the semigroup for any natural number, n. Mogbonju *et al.*(2019) undertook an extension to Mogbonju and Azeez (2018). Their study was on the signed singular self maps on the finite set $[n]$ ($SSing_n$). He found the cardinality of $SSing_n$ and the number of idempotents contained in it with respect to the collections of signed full transformation sets with composition not playing any role.

Tal *et al*. (2022) studied the work performed by the elements of $\widetilde{T}_n$. They characterised maps in $\widetilde{T}_n$ that attain maximum and minimum work as an extension to the work of Imam and Tal (2019) on the classical full transformations T_n.

In this paper, we extend Mogbonju and Azeez (2018) to the semigroups of signed order-decreasing, order-preserving and order-decreasing transformations.

METHODS AND MATERIALS

We present below existing definitions and results that are needed to achieve our aim. The results presented will serve as a building block upon which our main result shall hinge. All definitions can be found in any semigroup text like Howie (1995). Let SIO_n and SO_n denote the semigroups of signed injective order-preserving and signed order-preserving transformations on the set $[n]$ with $E(SIO_n)$ and $E(SO_n)$ as their corresponding semigroups of idempotents.

Definition 2.1: A Map $\alpha \in \widetilde{T}_n$ is an idempotent if $\alpha^2 = \alpha$. We denote the set of all idempotents of $\widetilde{T_n}$ by $E(\widetilde{T_n})$.

Definition 2.2: A transformation $\alpha \in \widetilde{T}_n$ is order-preserving if $\forall x, y \in [n],\ x \leq y \Longrightarrow \alpha x \leq \alpha y$

Definition 2.3: A transformation $\alpha \in \widetilde{T}_n$ is order-decreasing if $\forall x \in [n],\ \alpha x \leq x$

Definition 2.4: For a map $\alpha \in \widetilde{T}_n$, we define $h(\alpha) = |dom(\alpha)|$ and $f(\alpha) = |\{x \in dom(\alpha): x\alpha = x\}|$ to be the height and fix of α respectively.

Theorem 2.1 (Mogbonju and Azeez, 2018**):** Let $S = SIO_n$. Then, $|E(S)| = 2^n$.

Proposition 2.1 (Mogbonju *et al.*, 2019**):** Let $S = SO_n$. Then, $|E(S)| = \frac{1}{3n+1}\binom{4n}{n}$

Theorem 2.2 (Tal *et al.*, 2022)**:** Let $S = SSing_n$. If $\alpha \in$ S and $n \geq 0$, then $|E(S)| = \frac{n^2(n^2-1)}{3}$

For any map α: [n] → [n*], we shall use the famous notation below and for any map $\alpha \in \widetilde{T}_n$ we shall use the known standard notation for permutation to represent it.

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ \pm 1 & \pm 2 & \pm 3 & \ldots & \pm n \end{pmatrix}$$

RESULTS AND DISCUSSION

We present in this section the main findings of this work. But for the purpose of clarity, we note that if S is any of $SD_n, SOD_n,$ or SC_n, the signed transformations of $E(S)$ will be their corresponding idempotents.

Lemma 3.1: Let α ∈ SD_n. Then α is an idempotent element if the |im(α)| ≥ |f(α)|.

Proof: Let [n] = {1,2,3,...,n} and [n*] = $\{\pm 1, \pm 2, \pm 3, \ldots \pm n\}$. Suppose

$$\alpha_1 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_1 & a_2 & a_1 & a_2 \end{pmatrix} \in SD_n$$

We have that

$$\alpha_1{}^2 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_1 & a_1 & a_1 & a_1 \end{pmatrix} \neq \alpha_1$$

Thus, α_1 is not an idempotent element.

Again, let

$$\alpha_2 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_{-2} & a_2 & a_3 & a_3 \end{pmatrix}$$

Here, |im(α)| = 3, |f(α)| = 2 and

$$\alpha_2{}^2 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_{-2} & a_2 & a_3 & a_3 \end{pmatrix} = \alpha_2$$

Implies that |im(α)| > |f(α)|.

Furthermore, let

$$\alpha_3 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 & a_2 \end{pmatrix} \in SD_n,$$

where |(α)| = 3, |f(α)| = 3 and

$$\alpha_3{}^2 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 & a_2 \end{pmatrix} = \alpha_3.$$

It then implies that, |im(α)| = |f(α)| and so it follows that |im(α)| ≥ |f(α)| as required.

In the next result, we consider $|E(SOD_n)|$, the set of all idempotents of SOD_n, and count the number of idempotent in SOD_n.

Theorem 3.1: Let α ∈SD_n. Then

$$|E(SD_n)| = \begin{cases} 4^{n-1} & if\ n = 1,2,3. \\ 2^{n-4}(4^{n-1}) + 2(n+1) - 8(n-4)(3n-1) & if\ n \geq 4. \end{cases}$$

Proof: Suppose 1≤n≤ 3, then $E(SOD_n)= 4^{n-1}$ let α be a map in SOD_n. then α is an idempotent if $\alpha^2 = \alpha$.

For n= 1;

$$\begin{pmatrix} a_1 \\ a_1 \end{pmatrix}$$

For n = 2, we have;

$$\begin{pmatrix} a_1 & a_2 \\ a_1 & a_1 \end{pmatrix}, \begin{pmatrix} a_1 & a_2 \\ a_1 & a_{-1} \end{pmatrix}, \begin{pmatrix} a_1 & a_2 \\ a_1 & a_2 \end{pmatrix}, \begin{pmatrix} a_1 & a_2 \\ a_{-2} & a_2 \end{pmatrix}$$

Same result follows for values n= 3, we have 16 idempotent elements.

Similarly, in the next result we use mathematical induction to prove the case for n≥ 4 since $|E(SD_n)| = 2^{n-4}(4^{n-1}) + 2(n + 1) - 8(n - 4)(3n - 1)$, then certainly the result is true for n= 4.

Suppose n= k(by induction hypothesis), the result is true for all k, that is,

$$2^{K-4}(4^{K-1}) + 2(K + 1) - 8(K - 4)(3K - 1).$$

Furthermore, we assume n = k+1, we have,

$$2^{K-3}(4^{K}) + 2(K + 2) - 8(K - 3)(3K + 2)$$

Therefore, by induction the result holds for all n.

The result below counts the number of idempotent elements in SC_n.

Theorem 3.2: Let $\alpha \in E(SC_n)$. Then

$$|E(SC_n)| = \begin{cases} 1 & if n = 1 \\ 2^n if n is odd \\ 2^n - 1 & if n is even \end{cases}$$

Proof:

The result is obvious when $n = 1$.

Now, if n is odd, let us consider SC_n, a sub-semigroup of SOD_n. Thus $E(SC_n) \subseteq SC_n$ and an element of SC_n will be an idempotent if $|im(\alpha)| = |f(\alpha)| = 1$. Hence from the sequence $2, 8, 32, \ldots$ generated when n is odd for $E(SC_n)$ is likened to the power set which are number of subsets of a set, and of which they are special case of binomial theorem. We could observe also that this situation is same as most of idempotent elements of some sub-semigroups. Thus, $E(SC_n) = 2^n$ when n is odd.

Again, if n is even. Then any idempotent in $SC_n \subseteq SOD_n$ is such that $\alpha^2 = \alpha$. Observe that from the sequence generated, when n is even, the order of $E(SC_n)$ is one less than the power set. Hence by the result for an odd n, $E(SC_n) = 2^n - 1$, as required.

CONCLUSION

In this work, we employed a simple method of computing the number of idempotents elements of order-decreasing signed full transformation semigroups. This work answers the question of Mogbonju and Azeez (2018) on finding the n^{th} formula for signed full order-decreasing mapping. We recommend an extension of the idea presented in this work to the order-decreasing and order-preserving signed full transformation semigroup.

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