PHYSICAL SCIENCES

DIRECT NUMERICAL METHOD FOR GENERALIZED OPTIMAL CONTROL PROBLEMS CONSTRAINED WITH ORDINARY DIFFERENTIAL EQUATIONS.

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ABSTRACT
In this paper, we considered general class of continuous optimal control problem governed by Nth-order ordinary differential equations, in which the state and control variables are and column vectors respectively with corresponding matrix coefficients of dimension n×n, n×r. We adopt direct numerical method where the continuous optimal control problem is converted to a nonlinear programming problem via Augmented Langrangian which makes it amenable to optimization techniques (Conjugate Gradient Method). The result is compared with an existing method (exterior penalty method) and found to be more accurate.

Keywords: *Trapezoidal rule, Cranck-Nicholson, Augmented Lagrangian, Conjugate Gradient Method.*

INTRODUCTION

Optimal control problems are most often solved numerically because of the complexity of most applications; these numerical methods are dated back to the 1950's with the work of Bellman, (1966), Bellman and Dreyfus, (1959), Bellman *et al.*, (1963).

Optimal control problems governed by ordinary differential equations arise in a wide range of applications. Of special interest is the Linear Quadratic Optimal Control problem (LQOCP), which had been greatly studied Bertsekas, (1974), Ibiejugba and Onumanyi, (1984), Olotu and Olorunsola, (2006) due to its interesting features and its widerapplicability. Sargent, (2000) gave historical survey of optimal control and went on to review thedifferent approaches to the numerical solutions of optimal control problems.The function spacealgorithm for solving both continuous and discrete linear quadratic optimal control problems was given by Polak, (1971).

Most of the algorithms for solving unconstrained optimal control problems are based on a class of descent methods which traditionally have been the principal methods for solving unconstrained minimization problems. Efficient, within this class, are steepest descent (SD), Fletcher and Powell (1964), Klessig (1972) which had been classified as algorithms with no memory, and the Newton and quasi-Newton methods which update the hessian inverse of f(x).

In most applications, the conjugate gradient algorithm is more suitable when compared to other conjugate direction algorithms Hestenes and Stiefel, (1952). It totally outshinesthe steepest descent method, and compares more favourably with the Newton and quasi-Newton methods. For example, the Newton descent and Quasi-Newton descent method are not suitable for minimising the Rayleigh quotient associated with a matrix, since any attempt to approximate thehessian at the minimum is a singular matrix, Yang, (1989). Also when the dimension of the optimization variable is very large, most especially in optimal control, the conjugate gradient method is preferred.

Most research works in the field of unconstrained optimization concentrate their efforts on algorithms with inaccurate or no line search. This is due to the fact that the line search part is time-consuming. However, the reviewed literature was mainly analytical in approach and did not consider any direct method amenable to direct numerical algorithms, except for the recent publication by Olotu and Adekunle (2012), on the algorithm for a numerical solution to an optimal control problem governed by delay differential equation purely on the state variable with emphasis on vector-matrix coefficients.

This research seeks to address thedirect numerical approach using augmented lagrangian to solve this optimal control problem governed by ordinary differential equations with matrix coefficients using augmentedlagrangian to formulate the penalized matrix thereby rendering the nonlinear programming problem amenable to Conjugate Gradient method.

PROBLEM FORMATION

Consider the optimal control problem,

$$\min I(x,u) = \int_0^Z (x^T(t)Px(t) + x^T(t)Qu(t))dt \quad ....(2.1)$$

subject to $X(t) = Ax(t) + Bu(t),\ X(0) = X_0, 0 \le t \le Z.$ ....(2.2)

Unlike the indirect numerical method where the optimality condition is performed on the optimal control problem thereby resulting in boundary value problem, Olotu and Olorunsola (2006). The main idea in this work centers on the conversion of the continuous-time optimal control problem into a discretized Nonlinear Programming Problem(NLP) problem via the augmented multiplier method which makes it amenable to Optimization Techniques (Conjugate Gradient Method) so as to compute the near optimal control trajectories. This numerical result is then compared with that obtained from the use of exterior penalty method to form our Nonlinear programming Problem.

MATERIALS AND METHODS

Consider the optimal control problem,

$$\min I(x,u) = \int_0^Z (x^T(t)Px(t) + u^T(t)Qu(t))dt \quad ....(3.1)$$

subject to: $X(t) = Ax(t) + Bu(t),\ X(0) = X_0, 0 \le t \le Z.$ ....(3.2)

where $x(t) \in R^n, u(t) \in R^r, P_{n\times n}, Q_{n\times n}$ are symmetric positive definite and $A_{n\times n}, B_{n\times m}$ are not necessarily positive definite matrices.

We make equation (3.1) and (3.2) solvable by conjugate gradient method by replacing the constrained optimal control problem by appropriate approximate discretized control problem. We break the interval into equal sub-intervals with knots $t_0<t_1<t_2<......<t_w$ and say $\Delta t_i = 0.1$

Define $h = Z-0 / w$ ....(3.3)

Discretization of the performance index

We seek to discretize our cost functional using trapezoidal rule. We break the interval into equal subintervals such that

$$\int_0^Z (x^T(t)Px(t) + u^T(t)Qu(t))dt = h/2 \sum_{k=0}^{w} f(x_k) + f(x_{k-1}) + f(u_k) + f(u_{k-1}) \quad ......(3.1.1).$$

$$\min I = \frac{h}{2}\sum_{k=0}^{w} x_k^T Q(t_k)x_k + x_{k-1}^T Q(t_{k-1})x_{k-1} + x_k^T R(t_k)x_k + x_{k-1}^T R(t_{k-1})x_{k-1} \quad \text{............................ (3.1.2)}$$

$$\min I = \sum_{k=0}^{w} x_k^T \bar{Q}(t_k)x_k + x_{k-1}^T \bar{Q}(t_{k-1})x_{k-1} + x_k^T \bar{R}(t_k)x_k + x_{k-1}^T \bar{R}(t_{k-1})x_{k-1}, \quad \text{........................... (3.1.3)}$$

where $\bar{Q} = Q\frac{h}{2}$ and $\bar{R} = R\frac{h}{2}$

Similarly, we shall discretize our state equation using Crank-Nicholson Method

$$x_{k+1} - x_k = \frac{h}{2}\{f(x_{k+1}, u_{k+1}) + f(x_k, u_k)\} \quad \text{...................... (3.1.4)}$$

$$x_{k+1} - x_k = \frac{h}{2}\{Ax_{k+1} + Bu_{k+1} + Ax_k + Bu_k\}$$

$$(1 - A\frac{h}{2})x_{k+1} = (A\frac{h}{2} + 1)x_k + B\frac{h}{2}(u_{k+1} + u_k)$$

$$x_{k+1} = \bar{A}x_k + \bar{B}u_{k+1} + \bar{B}u_k, \quad \text{..(3.1.5)}$$

where $\bar{A} = (Ah + 2) * inv(2 - Ah)$ and $\bar{B} = inv(2 - Ah) * Bh$

Hence the discretised optimal control problem becomes,

$$\min I = \sum_{k=0}^{w} X_k^T \bar{Q}(t_k)X_k + X_{k-1}^T \bar{Q}(t_{k-1})x_{k-1} + x_k^T \bar{R}(t_k)x_k + x_{k-1}^T \bar{R}(t_{k-1})x_{k-1} \quad (3.4)$$

subject to $X_{k+1} = \bar{A}X_k + \bar{B}U_{k+1} + \bar{B}U_k$ ....(3.5)

By parameter optimization, Betts (2001), the discretised problem becomes a large sparsequadratic programming problem, written in matrix form as:

$$I(v) = v^T Tv + n \quad \text{.............(3.6)}$$

subject to

$$Hv = k \quad \text{........................(3.7)}$$

where T is a block diagonal matrix of order $(n+r)(w+r)$ with entries given by

$$T_{ii} = \begin{cases} 2\bar{Q}(t_i) & i = 1,2,3,\ldots,w-1 \\ \bar{Q}(t_i) & i = w \\ \bar{R}(t_i) & i = w+1 \\ 2\bar{R}(t_i) & i = w+2, w+3, \ldots, 2w \\ \bar{R}(t_i) & i = 2w+1 \end{cases} \quad \text{...........(3.8)}$$

$$v^T = (x_1^T, x_2^T, \ldots x_w^T, u_0^T, u_1^T, \ldots u_w^T) \quad \text{.............(3.9)}$$

where the i^{th} element corresponds to i^{th} block and $d = x_0\bar{Q}x_0$ such that matrix H is a block matrix of dimension $nw \times (n+r)w + r$. This can be written in matrix representation as

$$H = (E \mid F), \quad \text{................(3.10)}$$

where E is an $nw \times nw$ block matrix bidiagonal with principal block diagonal elements $[E]_{ii} = I_{n\times n} - \frac{h}{2}A(t_i)$ and lower principal diagonal $[E]_{ij} = -I_{n\times n} - \frac{h}{2}A(t_{i-1})$, for every i, j block such that $i = j+1$. The matrix F is an $nw \times (w+1)r$ block bidiagonal matrix with principal $[F]_{ii} = -\frac{h}{2}B(t_{i-1})$ and upper block principal diagonal elements $[F]_{ij} = -\frac{h}{2}B(t_i)$ for every i, j such that $j = i+1$. The column vector is of order with entries given by $[k]_{1:n,1} = \left(I_{n\times n} + \frac{h}{2}A(t_0)X_0\right)$ and $[k_{i,1}] = 0,\ i = n+1, n+2, \ldots nw$.

The unconstrained minimization problem by Augmented Lagrangian function is

$\min L_\rho = v^T T v + d + \lambda^T \left|Hv - k\right| + \frac{1}{\mu}\left\|Hv-k\right\|^2$ (3.11)

On expansion, we have

$\min L_\rho(v) = v^T T_\rho v + B^T v + C$ (3.12)

Equation (3.12) is the quadratic form representation for the unconstrained minimization problem, where $L_\rho(v)$ is the penalized lagrangian, ρ is the penalty parameter, the penalized matrix

$T_\rho = \left[V + \frac{1}{\mu}H^T H\right]$, $B^T = (\lambda^T H - \frac{2}{\mu}k^T H)$ and $C = (d - \lambda^T k + \frac{1}{\mu}k^T k)$.

The Operator $T_\rho = \left[V + \frac{1}{\mu}H^T H\right]$ is positive definite, see proof in Olotu and Akeremale, (2012)

NUMERICAL ALGORITHM

(1) Choose$Z_{0,0}$ $\epsilon \underline{R}^{(n+r)w+r}$,C>0 ,μ>0 ,λ>0,d0. set j=0

(2) Set i=0 and $p_0 = -g_0 = -\nabla L_p(Z_{0,0})$

(3) Compute $\alpha_i = \frac{(g_i^T g_i)}{p_i^T A p_i}$

(4) Set $Z_{(j,i+1)} = Z_{(j,i)} + \alpha_i p_i$

(5) Compute $\nabla L_p(Z_{j,i+1})$

(6) If $\nabla L_p(z_{(j,i+1)}) = 0$ *and* $JZ_{(j,i+1)} = K$, Stop. Else go to (7)

(7) If $\nabla Lp(z_{j,i+1}) \neq 0$, set $g_{i+1} = \nabla L_p(Z_{j,i+1})$

$p_{i+1} = -g_{i+1} + \gamma_i p_i$

$\gamma_i = \frac{(g^T_{i+1} g_{i+1})}{g_i^T g\_i}$

(8) Set $\boldsymbol{i}$=$\boldsymbol{i}$+1 and go to (3)

(9) Else, if $JZ_{j,i+1} \neq K$ *or* $JZ_{j,i+1} - K = 0,$ then

set $\mu_{k+1} = d\mu_k$

$\lambda_{j+1} = \lambda_j + \mu_j(JZ_j - K)$

(10) Set $\boldsymbol{j}$=$\boldsymbol{j}$+1 and go to step (2)

EXAMPLES AND PRESENTATION OF RESULTS

Example (1): consider the constrained optimal control problem

$$\min \quad I(x,u) = \int_0^1 (2x_1^2 + x_1 x_2 + x_2^2 + u_1^2 + \frac{1}{2}u_1 u_2 + u_2^2 \qquad \text{.......................(5.1)}$$

subject to: $\dot{x}_1 = 2x_1 - x_2 + u_1 + u_2$

$x_2 = x_1 - x_2 - u_1,$ ....(5.2) where $x(0) = 1_{2\times 1}$, It is clear that $P = \begin{pmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & \frac{1}{4} \\ \frac{1}{4} & 1 \end{pmatrix}$, $A = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$

By [19], the analytic objective value is $I = 2.6460$ and the objective value obtained using exterior penalty amenable to conjugate gradient method is $I = 2.6691$ why the objective value obtained using Augmented Lagrangian amenable to conjugate gradient method is $I = 2.6267$ as we will see in the Table 1.

Table 1. Comparison of results using existing scheme and the developed scheme.

Iterations	**Constraints Satisfaction**		**Objective Value**	
	DCAQP (2011)	New Scheme	DCAQP (2011)	New Scheme
1	0.8693E-1	0.4618	0.9158	-3.0329
2	0.1997E-1	0.9299E-1	2.0565	1.4277
3	0.2522E-2	0.8044E-2	2.5846	2.5591
4	0.2597E-3	0.4359E-3	2.6603	2.6248
5	0.2604E-4	0.2198E-4	2.6682	2.6266
6	0.2605E-5	0.1100E-5	2.6690	2.6267
7	0.2605E-6	0.1100E-5	2.6691	2.6267

Example (2): consider the constrained optimal control problem

$$\min \quad I(x,u) = \int_0^1 (x_1^2 + x_1 x_2 + x_2^2 + 2u_1^2 + 2u_1 u_2 + u_2^2)dt \quad \text{...(5.3)}$$

subject to:

$$\begin{aligned} \dot{x}_1 &= x_1 - x_2 + 2u_1 + u_2 \\ \dot{x}_2 &= x_1 + x_2 - u_2, \end{aligned} \quad \text{...(5.4)}$$

where $x(0) = 1_{2\times 1}$ $P = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, and $B = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$

By Olotu and Adekunle (2012), the analytic objective value is 2.5466 and objective value obtained by using exterior penalty amenable to conjugate gradient method is $I = 2.5656$ why the objective value obtained by using augmented lagrangian amenable to conjugate gradient method is $I = 2.5287$ as inTable 2 below.

Table 2. Comparison of results using existing scheme and the developed scheme.

Iterations	**Constraints Satisfaction**		**Objective Value**	
	DCAQP (2011)	New Scheme	DCAQP (2011)	New Scheme
1	0.8625E-1	0.4244	0.9008	-2.8797
2	0.2002E-1	0.9435E-1	2.0497	1.3483
3	0.2448E-2	0.7573E-2	2.4995	2.4694
4	0.2507E-3	0.4030E-3	2.5588	2.5271
5	0.2513E-4	0.2028E-4	2.5650	2.5287
6	0.2514E-5	0.1015E-5	2.5656	2.5287
7	0.2514E-6	0.1015E-5	2.5656	2.5287

By Olotu and Adekunle, (2012), the analytic objective value is 2.5466 and the objective value obtained by using exterior penalty amenable to conjugate gradient method is $I = 2.5656$ while the objective value obtained by using augmented lagrangian amenable to conjugate gradient method is $I = 2.5287$ as seen in the table above.

CONCLUSION

We have shown that generalized discrete optimal control problems with matrix coefficients can be solved directly via conjugate gradient method using exterior penalty method and Augmented Lagrangian method to construct the control operator (penalized matrix). However, it is observed that the new algorithm gives a better result in terms of accuracy, hence a better scheme. It is therefore recommended for generalized optimal control problems with delay-differential equations.

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