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# UNIQUENESS AND CONVERGENCE OF SOLUTION OF MULTI-TERM FRACTIONAL ORDER FREDHOLMINTEGRO-DIFFERENTIAL EQUATION

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## ABSTRACT
This paper focused on multi-term fractional order Fredholmintegro-differential equation which was transformed to integral equation by using Riemann-Liouville fractional integral. The uniqueness of solution of the multi-term fractional order Fredholmintegro-differential equation was proved using Banach contraction principle alongside the convergence of solution of the multi-term fractional order Fredholmintegro-differential equation, where Cauchy convergence criteria was used. Examples were given to prove the solvability of the multi-term fractional order Fredholmintegro-differential equation.

**Keywords:** Fractional order, uniqueness of solution, convergence, Fredholmintegro-differential equation.

## INTRODUCTION
Many physical events are better explained by fractional derivatives, because fractional operators take the evolution of the system into account (Almeida, Tavares and Torres, 2019). However, it is quite challenging to find analytical solutions for these fractional differential equations (FDEs). Therefore, numerical approximation plays a vital role in finding the approximate solution to these equations and as such, as it will be seen in the next section. Numerous scholars have created and introduced numerical approaches to find approximations to the solutions to this class of equations (Neimati, Lima and Torres, 2021).
Consequently, in the past few decades, applications of fractional calculus were reported in many branches of science, engineering and social sciences (Hilfer, 2000; Zabadal, Vilhena and Livotto, 2001; Oldham, 2010; Ertuk, Obidat and Momami; 2011; Yang, and Zhu, 2011; Magin, 2012; Fallahgoul, Focardi and Fabozzi, 2016; Zheng and Zhang, 2017; Mahmodov, 2017; Baleanu, Jajarmi and Hajipour, 2017; Singh, Kumar and Baleanu, 2017; Stoenoiu, Bolboaca and Jantschi, 2008; Hajipour, Jajarmi and Baleanu, 2018; Huang and Bae, 2018; Bulut *et al*., 2018; Baleanu and Lopez, 2019; Tarasov, 2020; Ming, Wang and Feckan, 2019; Mainadi, 2022). Fractional calculus has in the recent years attracted so much attention of researchers in various branches of mathematics and sciences as a whole. These researchers have created and studied the existence and uniqueness of solutions of different kinds of fractional differential equations such as Balachandran and Kiruthika(2011), Samko, Kilbas and Marichev(1993), Diethelm and Ford (2002), Yuste and Acedo (2005), Kilbas and Marzan (2005), Pilipovic and Stojanovic (2006), Agawal, Benchohra and Hamani (2010), Baleanu and Mustafa (2010), Tian and Bai (2010), Wei, Li and Che (2010), Anguraj, Karthikeyan and Trujillo (2011), Aghajani, Banas and Jalilian (2011), Idczak and Kamocki (2011), Kostic (2011), Agarwal and Ahmed (2011), Hu and Liu (2011), Aghajani, Jalilian and Trujillo (2012), Hamoud, Ghadle and Atshan (2019), Rui (2011) and Caballero, Harjani and Sadarangani (2011) and so many more.
In this study, we considered the multi-term fractional order Fredholmintegro-differential equation of the form

$$D^{\beta}\Psi(x)=\sum_{j=0}^{k}r_j(x)D^{\delta_j}\Psi(x)+g(x)+\int_0^1 K(\sigma,\tau)N(\Psi(\tau))d\tau \qquad (1)$$

subject to the initial condition

$$\Psi^{(n)}(0)=d_n,\ n=0,1,2,\dots,m-1, \qquad m-1<\delta_0<\delta_1<\cdots<\delta_j<\beta\le m,\ m\in \mathrm{N}, \qquad (2)$$

where $D$ is the differential operator defined in Caputo sense, $\Psi : Q\to \mathrm{R},\ Q=[0,1]$ is a continuous function which needs to be determined, $r_j$, $N : Q\to \mathrm{R}$ are given continuous functions, $K : Q\times Q\to \mathrm{R}$ is the kernel of integration which is also continuous, $N : \mathrm{R}\to \mathrm{R}$ is a Lipschitz function.

## MATERIALS AND METHODS
Definition (**Contraction Map** Rudin (1953)). Let $X$ be a metric space, a mapping $T : X\to X$ is a contraction if there exists $L\in[0,1)$ such that $\|Tx-Ty\|\le L\|x-y\|$ for all $x,y\in X$.
Definition (**Fixed Point** Zeidler (1986)). Given a map $T : A\to B$, every solution $y$ of the equation
$Ty=y$
is called a fixed point of $T$.
Definition **(Banach Contraction Principle** Zhou (2014)). Let $X$ be a complete metric space, then each

contraction mapping $T : Y \to Y$ has a unique fixed point $y$ of $T$ in $Y$; that is, $Ty = y$.

Definition (**Riemann-Liouville Fractional Integral,** Kilbas (2006)). Reimann-Liouville fractional integral of order $\beta$ of a function $\Psi$ is defined as

$$I^{\beta}\Psi(x)=\frac{1}{\Gamma(\beta)}\int_0^x (x-s)^{\beta-1}\Psi(s)ds, \qquad x>0,\ \beta\in \mathbb{R}^+, \qquad (3)$$

Where $\mathbb{R}^+$ is the set of positive real numbers.

Definition (**Riemann-Liouville Fractional Derivative** Kilbas (2006)). Reimann-Liouville fractional derivative of order $\beta$ of a function $\Psi$ is defined as

$$D^{\beta}\Psi(x)=D^m I^{m-\beta}\xi(x), \qquad m-1<\beta\le m,\ m\in \mathbb{N}$$
$$=\frac{d^m}{dt^m}\left(\frac{1}{\Gamma(m-\beta)}\int_0^x (x-s)^{m-\beta-1}\Psi(s)ds\right).$$

Definition (**Caputo Fractional Derivative** Kilbas (2006)). The fractional derivative of $\Psi(x)$ in the Caputo sense is defined by

$$D^{\beta}\Psi(x)=I^{m-\beta}D^m\Psi(x)$$
$$=\frac{1}{\Gamma(m-\beta)}\int_0^x (x-s)^{m-\beta-1}\frac{d^m\Psi(s)}{ds^m}ds, \quad m-1<\beta\le m. \qquad (4)$$

With the following properties

$I^{\beta}D^{\beta}\Psi(x)=\Psi(x)-\sum_{n=0}^{m-1}\frac{\Psi^{(n)}(0)}{n!}x^n$, $m-1<\beta\le m$,

$I^{\beta}D^{\gamma}\Psi(x)=I^{\beta-\gamma}\Psi(x)$ , $0<\gamma<\beta$ , and $m-1<\beta\le m,\ m\in\mathbb{N}$ ,

$I^{\alpha}\Psi(x)=\frac{x^{\alpha}}{\Gamma(\alpha+1)}$, where $y(x)=1,\ x\in[0,1]$.

## RESULT

In this paper, we denote by

$\|\cdot\|_{\infty}$ the sup norm on $C(Q,\mathbb{R})$, i.e for $r\in C(Q,\mathbb{R})$, $\|r\|_{\infty}=\sup_{x\in Q}|r(x)|$.

$\|\cdot\|_{\infty}$ the sup norm on $C(Q,\mathbb{R})$, i.e for $g\in C(Q,\mathbb{R})$, $\|g\|_{\infty}=\sup_{x\in Q}|g(x)|$.

We make the following hypotheses:

there exists a constant $\Psi>0$ such that for any $\Psi_1,\Psi_2\in C(Q,\mathbb{R})$ we have

$$|N(\Psi_1(x))-N(\Psi_2(x))|\le\Theta\|\Psi_1-\Psi_2\|_{\infty} \quad x\in[0,1]$$

there exists a constant $\Omega$ such that

$$\Omega=\sup_{\sigma\in[0,1]}\int_0^1|k(\sigma,\tau)|d\tau<\infty.$$

**Lemma 1**. Let $\Psi : Q\to\mathbb{R}$ and $g : Q\to\mathbb{R}$ be continuous functions. Then, a function $\Psi$ is a solution to the fractional integro-differential equation $(1)-(2)$ if, and only if,

$$\Psi(x)=\sum_{n=0}^{m-1}\frac{d_n}{n!}x^n+\sum_{j=0}^{k}\frac{r_j(x)}{\Gamma(\beta-\delta_j)}\int_0^x(x-\sigma)^{\beta-\delta_j-1}\Psi(\sigma)d\sigma$$
$$+\frac{1}{\Gamma(\beta)}\int_0^x(x-\sigma)^{\beta-1}g(\sigma)d\sigma \qquad (5)$$
$$+\frac{1}{\Gamma(\beta)}\int_0^x(x-\sigma)^{\beta-1}\left(\int_0^1K(\sigma,\tau)N(\Psi(\tau))d\tau\right)d\sigma.$$

Proof Using equation $(3)$ on equation $(1)$ and property (i), (ii) together with the initial condition $(2)$ we have,

$$I^{\beta}\left(D^{\beta}\Psi(x)\right)=I^{\beta}\left(\sum_{j=0}^{k}r_j(x)D^{\delta_j}\Psi(x)\right)+I^{\beta}\left(g(x)\right)+I^{\beta}\left(\int_0^1K(\sigma,\tau)N(\Psi(\tau))d\tau\right)$$
$$=\sum_{j=0}^{k}r_j(x)I^{\beta}\left(D^{\delta_j}\Psi(x)\right)+I^{\beta}\left(g(x)\right)+I^{\beta}\left(\int_0^1K(\sigma,\tau)N(\Psi(\tau))d\tau\right)$$
$$=\sum_{n=0}^{m-1}\frac{\Psi^{(n)}(0)}{n!}x^n+\sum_{j=0}^{k}\frac{1}{\Gamma(\beta-\delta_j)}\int_0^x(x-\sigma)^{\beta-\delta_j-1}r_j(\sigma)\Psi(\sigma)d\sigma$$
$$+\frac{1}{\Gamma(\beta)}\int_0^x(x-\sigma)^{\beta-1}g(\sigma)d\sigma$$
$$+\frac{1}{\Gamma(\beta)}\int_0^x(x-\sigma)^{\beta-1}\left(\int_0^1K(\sigma,\tau)N(\Psi(\tau))d\tau\right)d\sigma$$
$$=\sum_{n=0}^{m-1}\frac{d_n}{n!}x^n+\sum_{j=0}^{k}\frac{1}{\Gamma(\beta-\delta_j)}\int_0^x(x-\sigma)^{\beta-\delta_j-1}r_j(\sigma)\Psi(\sigma)d\sigma$$
$$+\frac{1}{\Gamma(\beta)}\int_0^x(x-\sigma)^{\beta-1}g(\sigma)d\sigma$$
$$+\frac{1}{\Gamma(\beta)}\int_0^x(x-\sigma)^{\beta-1}\left(\int_0^1K(\sigma,\tau)N(\Psi(\tau))d\tau\right)d\sigma.$$

Thus, $\Psi$ solves $(1)-(2)$ if, and only if, $\Psi$ solves $(5)$

**Theorem 1**. *(Uniqueness of Solution)* Assume that $(p_1)$ and $(p_2)$ holds, and if,

$$\left(\sum_{j=0}^{k}\frac{\|r_j\|_\infty}{\Gamma(\beta-\delta_j+1)}+\frac{\Theta\Omega}{\Gamma(\beta+1)}\right)<1, \tag{6}$$

then there is a unique solution $\Psi\in C(Q,\mathrm{R})$ to problem $(1)-(2)$.

Proof: Let $T$ be an operator such that $T:C(Q,\mathrm{R})\to C(Q,\mathrm{R})$ defined from equation $(5)$ as

$$\begin{aligned}(T\Psi)(x)=&\sum_{n=0}^{m-1}\frac{d_n}{n!}x^n+\sum_{j=0}^{k}\frac{1}{\Gamma(\beta-\delta_j)}\int_0^x(x-\sigma)^{\beta-\delta_j-1}r_j(\sigma)\Psi(\sigma)d\sigma\\ &+\frac{1}{\Gamma(\beta)}\int_0^x(x-\sigma)^{\beta-1}g(\sigma)d\sigma\\ &+\frac{1}{\Gamma(\beta)}\int_0^x(x-\sigma)^{\beta-1}\left(\int_0^1K(\sigma,\tau)N(\Psi(\tau))d\tau\right)d\sigma.\end{aligned} \tag{7}$$

The objective here is to apply Banach contraction principle. To do that, we will show that $T$ is a contraction.

First, we note that $T$ is well defined. Indeed, since $x\mapsto\sum_{n=0}^{m-1}\frac{d_n}{n!}x^n$, $x\mapsto\sum_{j=0}^{k}r_j(x)\left(I^{\beta-\delta_j}\Psi\right)(x)$, $x\mapsto\left(I^{\beta}g\right)(x)$, $x\mapsto I^{\beta}\left(\int_0^1K(\sigma,\tau)N(\Psi(\tau))d\tau\right)$ are continuous, the right hand of equation $(7)$ is well defined and $x\mapsto(T\Psi)(x)$ is continuous. Thus, for $\Psi\in C(Q,\mathrm{R})$, $T\Psi$ is also in $C(Q,\mathrm{R})$.

Let $\Psi_1,\Psi_2\in C(Q,\mathrm{R})$ and let $x\in[0,1]$. By definition of $T$ and denoting $U=(T\Psi_1)(x)-(T\Psi_2)(x)$, we have

$$\begin{aligned}|U|=&\Bigg|\sum_{j=0}^{k}\frac{1}{\Gamma(\beta-\delta_j)}\int_0^x(x-\sigma)^{\beta-\delta_j-1}r_j(\sigma)(\Psi_1(\sigma)-\Psi_2(\sigma))d\sigma\\ &+\frac{1}{\Gamma(\beta)}\int_0^x(x-\sigma)^{\beta-1}\left(\int_0^1K(\sigma,\tau)\left(N(\Psi_1(\tau))-N(\Psi_2(\tau))\right)d\tau\right)d\sigma\Bigg|\\ \le&\left|\sum_{j=0}^{k}\frac{1}{\Gamma(\beta-\delta_j)}\int_0^x(x-\sigma)^{\beta-\delta_j-1}r_j(\sigma)(\Psi_1(\sigma)-\Psi_2(\sigma))d\sigma\right|+\\ &\left|\frac{1}{\Gamma(\beta)}\int_0^x(x-\sigma)^{\beta-1}\left(\int_0^1K(\sigma,\tau)\left(N(\Psi_1(\tau))-N(\Psi_2(\tau))\right)d\tau\right)d\sigma\right|\\ \le&\sum_{j=0}^{k}\frac{1}{\Gamma(\beta-\delta_j)}\int_0^x(x-\sigma)^{\beta-\delta_j-1}|r_j(\sigma)||\Psi_1(\sigma)-\Psi_2(\sigma)|d\sigma+\\ &\frac{1}{\Gamma(\beta)}\int_0^x(x-\sigma)^{\beta-1}\left(\int_0^1|K(\sigma,\tau)||N(\Psi_1(\tau))-N(\Psi_2(\tau))|d\tau\right)d\sigma\\ \le&\sum_{j=0}^{k}\frac{\|r_j\|_\infty\|\Psi_1-\Psi_2\|_\infty}{\Gamma(\beta-\delta_j)}\int_0^x(x-\sigma)^{\beta-\delta_j-1}d\sigma+\frac{\Omega\Theta\|\Psi_1-\Psi_2\|_\infty}{\Gamma(\beta)}\int_0^x(x-\sigma)^{\beta-1}d\sigma\\ \le&\sum_{j=0}^{k}\frac{\|r_j\|_\infty\|\Psi_1-\Psi_2\|_\infty}{\Gamma(\beta-\delta_j+1)}x^{\beta-\delta_j}+\frac{\Omega\Theta\|\Psi_1-\Psi_2\|_\infty}{\Gamma(\beta+1)}x^{\beta}\end{aligned}$$

Thus,

$$\left\|T\Psi_1 - T\Psi_2\right\|_\infty \le \left(\sum_{j=0}^{k}\frac{\|r_j\|_\infty}{\Gamma(\beta-\delta_j+1)}+\frac{\Pi\Psi}{\Gamma(\beta+1)}\right)\left\|\Psi_1-\Psi_2\right\|_\infty,\ \text{for all } x\in[0,1].$$

We conclude that $T$ is a contraction since by equation (6), $\left(\sum_{j=0}^{k}\frac{\|r_j\|_\infty}{\Gamma(\beta-\delta_j+1)}+\frac{\Pi\Psi}{\Gamma(\beta+1)}\right)<1$. Hence, By Banach contraction principle, $T$ has a unique solution $\Psi$ in $C(Q,\mathrm{R})$.

**Theorem 2**. (*Convergence of Solution*). If the solution is convergent, then it converges to the exact solution of the Fredholm fractional integro-differential equation.

Proof Let $S_u, S_v$ be arbitrary partial sums with $v \le u$. We show that $S_u$ is a Cauchy sequence.

Let $S_u=\sum_{j=0}^{u}\Psi_j(x)$ and $S_v=\sum_{j=0}^{v}\Psi_j(x)$. Since $v\le u$, then, we have from equation $(5)$

$$\begin{aligned} S_u - S_v &= \sum_{j=v+1}^{u}\Psi_j(x) \\ &= \sum_{j=0}^{k}\frac{1}{\Gamma(\beta-\delta_j)}\int_0^x (x-\sigma)^{\beta-\delta_j-1} r_j(\sigma)\sum_{j=v+1}^{u}\Psi_j(\sigma)\,d\sigma \\ &\quad+\frac{1}{\Gamma(\beta)}\int_0^x (x-\sigma)^{\beta-1}\left(\int_0^1 K(\sigma,\tau)N\left(\sum_{j=v+1}^{u}\Psi_j(\tau)\right)d\tau\right)d\sigma. \end{aligned} \tag{8}$$

Let $N\left(\sum_{j=v+1}^{u}\Psi_j(\tau)\right)=\sum_{j=v+1}^{u}H_j(\tau)$, then equation $(8)$ becomes

$$\begin{aligned} |S_u - S_v| &= \left|\sum_{j=v+1}^{u}\Psi_j(x)\right| \\ &= \left|\sum_{j=0}^{k}\frac{1}{\Gamma(\beta-\delta_j)}\int_0^x (x-\sigma)^{\beta-\delta_j-1} r_j(\sigma)\sum_{j=v+1}^{u}\Psi_j(\sigma)\,d\sigma\right. \\ &\quad\left.+\frac{1}{\Gamma(\beta)}\int_0^x (x-\sigma)^{\beta-1}\left(\int_0^1 K(\sigma,\tau)N\left(\sum_{j=v+1}^{u}\Psi_j(\tau)\right)d\tau\right)d\sigma\right|. \end{aligned} \tag{9}$$

If, we let $\sum_{j=v}^{u-1}\Psi_j(x)=S_{u-1}-S_{v-1}$, $\sum_{j=v}^{u-1}H_j(t)=N(S_{u-1})-N(S_{v-1})$ in equation $(9)$, then we have

$$\begin{aligned} \|S_u - S_v\|_\infty &\le \max_{\forall x\in Q}\left(\left|\sum_{j=0}^{k}\frac{1}{\Gamma(\beta-\delta_j)}\int_0^x (x-\sigma)^{\beta-\delta_j-1} r_j(\sigma)(S_{u-1}-S_{v-1})\,d\sigma\right.\right. \\ &\quad\left.\left.+\frac{1}{\Gamma(\beta)}\int_0^x (x-\sigma)^{\beta-1}\left(\int_0^1 K(\sigma,\tau)\left(N(S_{u-1})-N(S_{v-1})\right)d\tau\right)d\sigma\right|\right) \\ &\le \max_{\forall x\in Q}\left(\sum_{j=0}^{k}\frac{1}{\Gamma(\beta-\delta_j)}\int_0^x (x-\sigma)^{\beta-\delta_j-1}|r_j(\sigma)||S_{u-1}-S_{v-1}|\,d\sigma\right. \\ &\quad\left.+\frac{1}{\Gamma(\beta)}\int_0^x (x-\sigma)^{\beta-1}\left(\int_0^1 |K(\sigma,\tau)||N(S_{u-1})-N(S_{v-1})|\,d\tau\right)d\sigma\right) \\ &\le \sum_{j=0}^{k}\frac{\|r_j\|_\infty\|S_{u-1}-S_{v-1}\|_\infty}{\Gamma(\beta-\delta_j)}\int_0^x (x-\sigma)^{\beta-\delta_j-1}\,d\sigma \\ &\quad+\frac{\Pi\Psi\|S_{u-1}-S_{v-1}\|_\infty}{\Gamma(\beta)}\int_0^x (x-\sigma)^{\beta-1}\,d\sigma \\ &\le \left(\sum_{j=0}^{k}\frac{\|r_j\|_\infty}{\Gamma(\beta-\delta_j+1)}+\frac{\Omega\Theta}{\Gamma(\beta+1)}\right)\|S_{u-1}-S_{v-1}\|_\infty. \end{aligned}$$

Thus,

$$\left\|S_u - S_v\right\|_\infty \le \varphi \left\|S_{u-1} - S_{v-1}\right\|_\infty . \tag{10}$$

where $\varphi = \sum_{j=0}^{k} \frac{\|r_j\|_\infty}{\Gamma(\beta-\delta_j+1)} + \frac{\Omega\Theta}{\Gamma(\beta+1)}$.

Observe that, from equation $(10)$

$$\left\|S_u - S_v\right\|_\infty \le \varphi \left\|S_{u-1} - S_{v-1}\right\|_\infty \le \varphi \left\|S_{u-2} - S_{v-2}\right\|_\infty \le \ldots \le \varphi \left\|S_1 - S_0\right\|_\infty$$

Also from equation $(10)$

$$\left\|S_{u-1} - S_{v-1}\right\|_\infty \le \varphi \left\|S_{u-2} - S_{v-2}\right\|_\infty , \ \left\|S_{u-2} - S_{v-2}\right\|_\infty \le \varphi \left\|S_{u-3} - S_{v-3}\right\|_\infty , \ldots$$

Therefore,

$$\left\|S_u - S_v\right\|_\infty \le \varphi \left\|S_{u-1} - S_{v-1}\right\|_\infty \le \varphi^2 \left\|S_{u-2} - S_{v-2}\right\|_\infty \le \ldots \le \varphi^v \left\|S_1 - S_0\right\|_\infty \tag{11}$$

Let $u = v+1,$ accordingly in equation $(11)$ then we have

$$\left\|S_u - S_v\right\|_\infty \le \varphi \left\|S_v - S_{v-1}\right\|_\infty \le \varphi^2 \left\|S_{v-1} - S_{v-2}\right\|_\infty \le \ldots \le \varphi^v \left\|S_1 - S_0\right\|_\infty .$$

That is,

$$\left\|S_p - S_q\right\|_\infty \le \varphi^v \left\|S_1 - S_0\right\|_\infty \tag{12}$$

$\left\|S_u - S_v\right\|_\infty$ can be written as follows

$$\begin{aligned} \left\|S_u - S_v\right\|_\infty &= \| S_{v+1} - S_v + S_{v+2} - S_{v+1} + S_{v+3} - S_{v+2} + S_{v+4} - S_{v+3} \\ &\quad + \cdots + S_{v+(u-v-2)} - S_{v+(u-v-1)} + S_{v+(u-v)} - S_{v+(u-v-1)} \|_\infty \\ &= \| S_{v+1} - S_v + S_{v+2} - S_{v+1} + S_{v+3} - S_{v+2} + S_{v+4} - S_{v+3} \\ &\quad + \cdots + S_{u-2} - S_{u-1} + S_u - S_{u-1} \|_\infty \\ &\le \left\|S_{v+1} - S_v\right\|_\infty + \left\|S_{v+2} - S_{v+1}\right\|_\infty + \left\|S_{v+3} - S_{v+2}\right\|_\infty + \\ &\quad \left\|S_{v+4} - S_{v+3}\right\|_\infty + \cdots + \left\|S_{u-2} - S_{u-1}\right\|_\infty + \left\|S_u - S_{u-1}\right\|_\infty . \end{aligned} \tag{13}$$

From equation $(12)$, let $u = v+1,$ then

$$\begin{aligned} \left\|S_{v+1} - S_v\right\|_\infty &\le \varphi^v \left\|S_1 - S_0\right\|_\infty \\ \left\|S_{v+2} - S_{v+1}\right\|_\infty &\le \varphi^{v+1} \left\|S_1 - S_0\right\|_\infty \\ \left\|S_{v+3} - S_{v+2}\right\|_\infty &\le \varphi^{v+2} \left\|S_1 - S_0\right\|_\infty \\ &\vdots \\ \left\|S_u - S_{u-1}\right\|_\infty &\le \varphi^{u-1} \left\|S_1 - S_0\right\|_\infty . \end{aligned}$$

Therefore, equation $(13)$ can be written as

$$\begin{aligned} \left\|S_u - S_v\right\|_\infty &\le \left(\varphi^v + \varphi^{v+1} + \varphi^{v+2} + \cdots + \varphi^{u-1}\right) \left\|S_1 - S_0\right\|_\infty \\ &= \varphi^v \left(1 + \varphi^1 + \varphi^2 + \cdots + \varphi^{u-v-1}\right) \left\|S_1 - S_0\right\|_\infty . \end{aligned}$$

By geometric series, this implies

$$\left\|S_u - S_v\right\|_\infty \le \varphi^v \left(\frac{1-\varphi^{u-v}}{1-\varphi}\right) \left\|S_1 - S_0\right\|_\infty$$

since $0 < \varphi < 1,$ this means $1 - \varphi^{u-v} < 1,$ then

$$\left\|S_u - S_v\right\|_\infty \le \frac{\varphi^v}{1-\varphi} \left\|\Psi_1\right\|_\infty .$$

But $\left|\Psi_1(x)\right|<\infty$ and $\lim_{v\to\infty}\frac{\varphi^v}{1-\varphi}=0,$ since $\varphi^v\to 0$ as $v\to\infty.$ Therefore, $\left\|S_u-S_v\right\|_\infty\to 0$ as $v\to\infty.$ We conclude that $S_u$ is a Cauchy sequence in $C[0,1].$ Therefore, $\lim_{n\to\infty}\Psi_n=\Psi.$ Thus, the solution is convergent.

**Examples**

**Example 1** (Ghomanjani (2020)). Consider the fractional order integro-differential equation of the Fredholm type

$$D^{\frac{5}{3}}\xi(x)=\frac{3\sqrt{3}\Gamma\left(\frac{2}{3}\right)}{\pi}x^{\frac{1}{3}}-\frac{x^2}{5}-\frac{x}{4}+\int_0^1\left(x\tau+x^2\tau^2\right)\xi(\tau)\,d\tau, \qquad (14)$$

Subject to $\xi(0)=\xi'(0)=0$ with exact solution as $\xi(x)=x^2$ .

**Solution**

From Theorem 1, and denoting $U=(T\Psi_1)(x)-(T\Psi_2)(x)$ as before, we have

$$\begin{aligned}
|U|&=\left|\frac{1}{\Gamma\left(\frac{5}{3}\right)}\int_0^x(x-\sigma)^{\frac{5}{3}-1}\left(\int_0^1\left(\sigma\tau+\sigma^2\tau^2\right)\left(\Psi_2(\tau)-\Psi_1(\tau)\right)d\tau\right)d\sigma\right|\\
&\le\frac{1}{\Gamma\left(\frac{5}{3}\right)}\int_0^x(x-\sigma)^{\frac{5}{3}-1}\left(\int_0^1\left|\sigma\tau+\sigma^2\tau^2\right|\left|\Psi_2(\tau)-\Psi_1(\tau)\right|d\tau\right)d\sigma\\
&\le\frac{\left\|\Psi_2-\Psi_1\right\|_\infty}{\Gamma\left(\frac{5}{3}\right)}\int_0^x(x-\sigma)^{\frac{5}{3}-1}\left(\int_0^1\left|\sigma\tau+\sigma^2\tau^2\right|d\tau\right)d\sigma\\
&=\frac{\left\|\Psi_2-\Psi_1\right\|_\infty}{\Gamma\left(\frac{5}{3}\right)}\int_0^x(x-\sigma)^{\frac{5}{3}-1}\left(\frac{\sigma}{2}+\frac{\sigma^2}{3}\right)d\sigma\\
&\le\left(\frac{\Gamma(2)x^{\frac{8}{3}}}{2\Gamma\left(\frac{11}{3}\right)}+\frac{\Gamma(3)x^{\frac{11}{3}}}{3\Gamma\left(\frac{14}{3}\right)}\right)\left\|\Psi_2-\Psi_1\right\|_\infty.
\end{aligned}$$

Thus,

$$\left\|T\Psi_2-T\Psi_1\right\|_\infty\le(0.16994)\left\|\Psi_2-\Psi_1\right\|_\infty.$$

We see that $0.16994<1,$ Hence, the example above satisfies the condition of $Theorem\ 1$ and therefore, unique solution exists.

**Example 2** $Rostamy\,(2013)$ Consider the multi-term fractional order integro-differential equation

$$D^2\Psi(x)-x^2D^{\frac{3}{2}}\Psi(x)-\sqrt{x}D^{\frac{1}{2}}\Psi(x)-\sqrt[3]{x}\Psi(x)+\lambda\int_0^1 xt^2\Psi(t)\,dt=6\sqrt{\pi}x-8\sqrt{x^7}-\frac{16}{5}x^3-\sqrt[3]{x^{10}}\sqrt{\pi}. \qquad (15)$$

Subject to $\Psi(0)=\Psi'(0)=0$ with exact solution $\Psi(x)=\sqrt{\pi}x^3$ and $\lambda=0$ .

**Solution:** Equation $(15)$ can be written as

$$\left|\left(T\Psi_2\right)(x)-\left(T\Psi_1\right)(x)\right| \le \frac{1}{\Gamma(2)\Gamma(1.5)}\int_0^x (x-\sigma)\sigma^{\frac{5}{2}}\left|\Psi_2(\sigma)-\Psi_1(\sigma)\right|d\sigma$$

$$+\frac{1}{\Gamma(2)\Gamma(2.5)}\int_0^x (x-\sigma)\sigma^{2}\left|\Psi_2(\sigma)-\Psi_1(\sigma)\right|d\sigma$$

$$+\frac{1}{\Gamma(2)}\int_0^x (x-\sigma)\sigma^{\frac{1}{3}}\left|\Psi_2(\sigma)-\Psi_1(\sigma)\right|d\sigma$$

$$\le \frac{\left\|\Psi_2-\Psi_1\right\|_\infty}{\Gamma(2)\Gamma(1.5)}\int_0^x (x-\sigma)\sigma^{\frac{5}{2}}d\sigma$$

$$+\frac{\left\|\Psi_2-\Psi_1\right\|_\infty}{\Gamma(2)\Gamma(2.5)}\int_0^x (x-\sigma)\sigma^{2}d\sigma$$

$$+\frac{\left\|\Psi_2-\Psi_1\right\|_\infty}{\Gamma(2)}\int_0^x (x-\sigma)\sigma^{\frac{1}{3}}d\sigma$$

$$=\left(\frac{\Gamma(3.5)x^{4.5}}{\Gamma(5.5)\Gamma(1.5)}+\frac{\Gamma(3)x^{4}}{\Gamma(2.5)\Gamma(5)}+\frac{\Gamma\left(\frac{4}{3}\right)x^{\frac{7}{3}}}{\Gamma\left(\frac{10}{3}\right)}\right)\left\|\Psi_2-\Psi_1\right\|_\infty .$$

Thus,

$$\left\|T\Psi_2-T\Psi_1\right\|_\infty \le (0.45576)\left\|\Psi_2-\Psi_1\right\|_\infty .$$

Since $0.45576<1$, we say that example 2 satisfies the condition of the *Theorem 1*.

**CONCLUSION**

This paper focuses on multi-term fractional order Fredholmintegro-differential equation which was transformed to integral equation by using Riemann-Liouville fractional integral. The study of the uniqueness of solution of the multi-term fractional order Fredholmintegro-differential equation was proved alongside the convergence of solution of the multi-term fractional order Fredholmintegro-differential equation. Examples were given to prove the solvability of the multi-term fractional order Fredholmintegro-differential equation.

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