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SOLUTIONS OF SOME LINEAR VOLTERRA INTEGRAL EQUATIONS BY LAPLACE TRANSFORM METHOD

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ABSTRACT

This work provides solutions to some continuous and weakly singular linear Volterra integral equations of thesecond kind by Laplace transform method. Using basic definitions, results and fundamental theorems, Laplace transform method gives an efficient and remarkable performance.Test problems are presented to show the efficiency and reliability of the method.

Keywords: *Volterra Integral equations, continuous and weakly singular Kernels, Laplace method*

INTRODUCTION

Vito Volterra, an Italian Mathematician(1860-1940) was the first Mathematician to work on Integral equations which are now called Volterra Integral equations. In his work, he came out with useful fundamental theorems that are still useful till today. Mathematical formulations of physical phenomena from fluid dynamics, biological models and chemical kinetics also resulted in integral and integro-differential equations.

Babolian and Salimi (2008) in their work used operational matrices of piecewise constant orthogonal functions on the interval (0,1) to solve Volterra integral and Integro-differntial equations of convolution type. Baratella and Orsi (2004) used a new approach to obtain the numerical solution of weakly singular Volterra integral equations. Cooke (1976), in his work on epidemic equation with immigration formulated a model of single-specie population growth in which there is immigration into the population at any prescribed rate and age distribution. His model resulted in non linear-non-homogeneous integral equation with delay are shown to be model forgrowth of certain epidemics. Fazhan and Feng(2010), also worked on product integration of Volterra integral equations of the second kind with weakly singular kernels. He introduced a new numerical approach for solving Volterra equation of the second kind when the kernel contains a mild singularity and gave a convergence resultand presented numerical examples which show the performance and efficiency of the method. Jafar *et al.,* (2016) in their work converted a nonlinear weakly singular Volterra integral equation to a non-singular one by new fractional–order Legendre functions and solving the problem by Legendre Pseudospectral method. In another work by Jie Sheng *et al.,* (2010), a generalized Jacobi spectral-Galerkin method for the nonlinear Volterra integral equations with weakly singular kernel and established the existence and uniqueness of numerical solutions and characterize the convergence of the proposed method under reasonable assumptions on the nonlinearity. In another work by Odibat (2008), he solved Volterra equations with separable kernels by using differential transform method. Approximate solutions of the equations are calculated in form of series with easily computable terms. The exact solutions of the linear and non linear integral equations are investigated and the results illustrate the reliability and performance of the method. Orsi (1196) used product integration to solve Volterra integral equations of the second kind with weakly singular kernel by transforming the unknown function to obtain another weakly singular equation whose solution is smooth when solved by standard product integration method. Xuequin and Sixing (2012) used producing kernel method (RKM) and Adomia decomposition method (ADM) to solve nth order nonlinear weakly singular Volterra Integro differential equations. In this work, solutions of some continuous and weakly singular linear Volterra integral equations of the second kind are provided using Laplace transform method.

MATERIALS AND METHODS

Definition 2.1 [3]

A linearVolterra integral equation of the second kind is a functional equation of the form

$$u(t) = f(t) + \int_0^{\infty} K(t,s)u(s)ds, t \in I = [0,T]$$ ..(2.1.1)

wheref(t),K(t,s)are given functions and u(t)is an unknown function. The function K(t,s) is calledthe kernel of the Volterra integral equation.

Definition 2.2 [5]

The Laplace transform of a function f(t), is defined by

$$\mathcal{L}[f(t) = F(s) = \int_0^{\infty} e^{-st} f(t)dt$$ ....(2.2.1)

whenever the integral in (2.2.1) exists (converges) or does not exist(diverges); s is real, t≥0, and Lis theLaplace transform operator.

Definition 2.3 [5,6]

Let f(t) and g(t)be piecewise continuous functions on [0,∞],then the convolution integral of f(t) and g(t) is

$$f(t) * g(t) = \int_0^{\infty} f(t-s)g(s)ds$$ ....(2.3.1)

Note that

$$f(t) * g(t) = \int_0^{\infty} f(t-s)g(s)ds$$

$$= \int_0^{\infty} g(t-s)f(s)ds = g(t) * f(t)$$ ....(2.3.2)

Now if

$\mathcal{L}$[f(t)]=F(s) and $\mathcal{L}$[g(t)]=G(s) then

$\mathcal{L}$[f(t)*g(t)]=F(s)*G(s)=$\mathcal{L}$[f(t)]*$\mathcal{L}$[g(t)]

and

$$\mathcal{L}^{-1}[F(s) * G(s))] = \int_0^{t} f(x)g(t-x)dx = f(t) * g(t)$$ (2.3.3)

where $x = t - s$

Volterra Theorem 2.4 [3,8]

Assume that the kernel K(t,s) of the linear Volterra integral equation (2.1.1) is continuous on

D:={(t,s)≤s≤t≤T}. Then, for any function g(t) that is continuous on[0,∞], the Volterra integralequation possesses a unique solution $u(t) \in C(I)$which can be written in the form.

$$u(t) = f(t) + \int_0^{\infty} R(t,s) f(s) ds, t \in I$$

for some $R \in C(D)$.The function R=R(t,s) is called the Resolvent kernel of the given kernel K(t,s)

Definition 2.5 [3,8]

A Volterra integral equation with weakly singular kernels is defined as

$$u(t) = f(t) + \int_0^{\infty} K_\alpha(t,s) u(s) ds, 0 < \alpha < 1 \qquad (2.5.1)$$

The kernel $K_\alpha=(t-s)^{-\alpha}$ K(t,s)is a weakly singular kernel called the Integrable kernel. It is bounded whent=sbut its integral over any bounded interval [*0,T*] is finite. Also, K(s,t) is continuous on D and satisfies $K(t,t) \neq 0, t \in I$

Remark 2.6[3,8]

Volterra integral equations with weakly singular kernels K_α (t,s)=$(t,s)^{-\alpha}$ K(t,s,),0<α<1are called Abel Integral equations. It is named after a Norwegian Mathematician, NielsHenrik Abel(1802-1829) who was the first Mathematician to study such integral.

Volterra Theorem 2.7 [3,8]

Let 0<α<1 and assume that $f \in C^d(I), K \in C^d(D)$for some d≥0

If d=0, then the Volterra integral equation (2.5.1), $t \in I$, possesses a unique solution $u(t) \in C(I)$ which can be written in the form

$$u(t) = f(t) + \int_0^{\infty} R_\alpha(t,s) f(s) ds, t \in I := [0,T] \qquad ...(2.7.1)$$

where the resolvent kernel R_α (t,s)of the kernel K_α(t,s) is of the form

R_α (t,s)=$(t-s)^{-\alpha}$ Q_α (t,s), whereQ_α (t,s)is continuous on D.

Ifd≥1, then every non trivial solution has the property that $u(t) \in C^1(I)$ (I)and as t→0^+,the solutionbehaves like $u'(t) \sim Ct^{-\alpha}$

Corollary 2.8 [13,14]

Assume that $K \in C(I)$, then for any given $f \in C(I)$, the Volterra integral equation

$$u(t) = f(t) + \int_0^{\infty} K(t-s) f(s) ds, t \in I \qquad .....(2.8.1)$$

Possesses a unique solution given by

$$u(t) = f(t) + \int_0^{\infty} r(t-s) f(s) ds, t \in I \qquad ......(2.8.2)$$

where the resolvent kernel R(t,s) of K(t,s)=K(t-s) has the convolution form

R(*t,s*)=r(*t-s*) ....(2.8.3)

Method of Solution

Consider a linear Volterra Integral equation of the second kind.

$$a_i u + \sum_{i=0}^{\kappa} b_i \int_a^t K_i(t,s) u(s) ds + f(t) = 0 \qquad ....(3.1)$$

wheref(t) and the kernel, K(t,s) of the integral are defined on the triangle a≤u≤t≤b,u is to be determined with prescribed conditions and a_i,b_i are constants. WhenK(t,s) and f(t) are continuous, the integral part of (3.1) has a unique continuous solution as in Theorem (2.4). Using definition (2.3), we obtain

$$a_i \mathcal{L}[u] + \sum_{i=0}^{\kappa} b_i \mathcal{L}[K_i(t,s) u(s) ds] + \mathcal{L}[f(t)] = 0 \qquad ..(3.2)$$

$$a_i \bar{u}[s] + \sum_{i=0}^{\kappa} b_i \bar{K}_i(s) * \frac{d^i \bar{y}(t,s)}{dt^i}] + F(s)] = 0 \qquad ......(3.3)$$

Replace $\bar{u}[s]$ by f(s) and solve the algebraic equations for the transform. Then, take the inverse Laplace transform to obtain the solution of (3.1)

Test Problem1: Solve the linear Volterra Integral Equation with continuous kernel

$$f(t) = e^t + 2\int_0^t \cos(x-t) f(t) dt$$

$$\mathcal{L}\left[f(t)\right] = \mathcal{L}[e^t + 2\int_0^t \cos(x-t) f(t) dt]\Big]$$

$$\bar{y}[s] = \frac{1}{s-1} + \frac{2f(s) * s}{s^2+1} \qquad ....(3.4)$$

Replace $\bar{y}[s]$ *by* $F(s)$in (3.4) to obtain

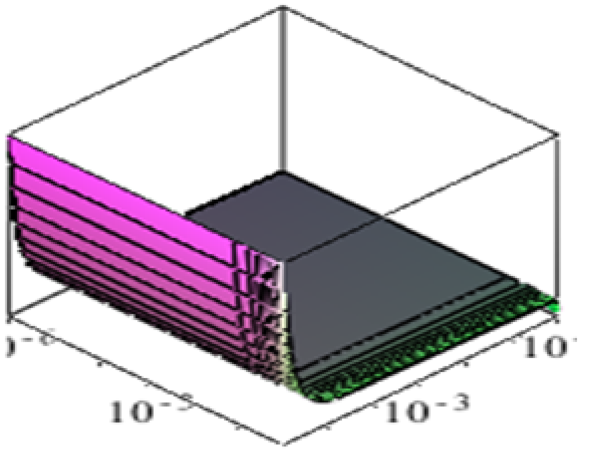
$$F(s) = \frac{1}{s-1} + \frac{2f(s) * s}{s^2+1} \qquad ....(3.5)$$

Solving (3.5) gives

$$F(s) = \frac{s^2+1}{(s-1)(s^2-2s+1)} \qquad ....(3.6)$$

Using definition (2.4) to get the inverse Laplace transform $\mathcal{L}^{-1}$ of (3.6) as

$$u(t) = e^t(t+1)^2 \qquad ....(3.7)$$



Test Problem2: Solve the linearVolterra Integral Equation with continuous kernel

$$f(t) = \cos(t) - \int_0^x (x-t)\cos(x-t)f(t)dt$$

Taking the Laplace transform of both sides,

$$\mathcal{L}\left[f(t)\right] = \mathcal{L}[\cos(t) - \int_0^t (x-t)\cos(x-t)f(t)dt]$$ \

$$= \mathcal{L}[\cos(t) - t * \cos(t) * f(s)\Big|$$

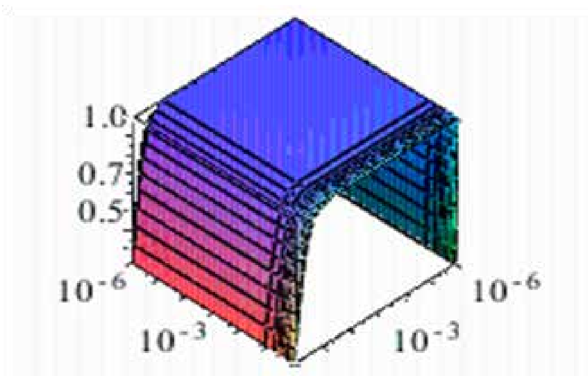
$$F(s) = \frac{s}{s^2+1} - \frac{f(s)(s^2-1)}{(s^2+1)^2}$$ ....(3.7)

Solving (3.7) to obtain

$$F(s) = \frac{s^2+1}{(s(3+s^2))}$$ ....(3.8)

Using definition (2.4) to get the inverse of $\mathcal{L}^{-1}$of (3.8) as

$$u(t) = \frac{2}{3}\cos(\sqrt{3t}) + \frac{1}{3}$$ ....(3.9)



Test Problem 3: Solve the weakly singular Kernel Abel's Equation

$$\int_0^x \frac{f(t)}{(x-t)^{1/3}} dt = \frac{\pi}{2}t + t^2$$

Taking the Laplace transform of both sides

$$\mathcal{L}\left[t^{-\frac{1}{3}} * f(s)\right] = \mathcal{L}[\frac{\pi}{2}t + t^2\Big]$$ ....(3.10)

$$\frac{f(s)\Gamma(\frac{2}{3})}{s^{\frac{2}{3}}} = \frac{\pi}{2s^2} + \frac{2}{s^3}$$,............................(3.11)

Solving (3.11) to obtain

$$F(s) = \frac{1}{2}\frac{\pi s + 4}{s^{\frac{7}{3}}\Gamma(\frac{2}{3})}$$ ....(3.12)

Using definition (2.4) to get the inverse of $\mathcal{L}^{-1}$of (3.12) as

$$u(t) = \frac{3}{4}\frac{t^{1/3}}{\pi}\sqrt{3}(3t+\pi)$$ ....(3.13)

RESULTS AND DISCUSSION

In this work, Laplace transform method was successfully applied to find the exact solutions of linearVolterra equations with continuous and weakly singular kernels. Some test problems are presented and ourresults show a remarkable performance of the method. This shows the reliability and efficiency of the method.

CONCLUSION

Laplace transform method can be used to provide exact solutions to linear Volterra integral equations of the second kind with continuous and weakly singular kernels. The method gives an efficient and remarkable performance. Test problems are also presented to show the efficiency and reliability of the method.

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CONFLICT OF INTEREST

The authors declare that there is no conflict of interest

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