FULafia Journal of Science & Technology
ISSN (print) 2449-0954 \ ISSN (online) 2636-4972
https://lafiascijournals.org.ng/index.php/fjst
Sponsored By TERTIARY EDUCATION TRUST FUND
Manuscript => Received: Dec., 2023; Accepted: March, 2024; Published: April, 2024
https://doi.org/10.62050/fjst2024.v8n1.231

EXISTENCE AND UNIQUENESS OF SOLUTIONS TOA SECOND ORDER DIFFERENCE EQUATION OF ACCRETIVE TYPE IN 2-BANACH SPACES

G. T. Gyegwe* and I. G. Bassi
Department of Mathematics, Federal University of Lafia, Nasarawa State, Nigeria
*Corresponding email: ggyegwe@gmail.com

ABSTRACT
In this paper, we investigate the existence and uniqueness of solutions of a homogeneous second order difference inclusion of accretive type in 2-Banach spaces using expansive mappings and 2-Banach contraction mapping.

Keywords: Accretive operator, m-accretive operator, 2-Normed space, 2-Banach space, expansive mappings.

INTRODUCTION

The study of accretive operators in Banach spaces has been done by many researchers. Among this area of study is the homogeneous second-order difference equation of accretive type in Banach spaces of the form

$$\begin{cases} u_{j+1} - (1+\beta_j)u_j + \beta_j u_{j-1} \in c_j A u_{j,} \ j \geq 1 \\ \\ u_0 = g, \sup\{\| \mathrm{u_j} \| : \mathrm{j} \geq 0\} < \infty, \end{cases} \qquad (1.1)$$

Where A is a nonlinear m-accretive operator in real Banach space $(B, \|.\|)$, $c_j > 0$, and β_j (Rouhani *et al.* 2019).

Lakshmikantham and Trigiante (2003) posited that the concept of computing by recursion can be traced back to the time counting started. Its primitive form emanated from the efforts of the Babylonians as early as 2000 B.C. Since then, many mathematicians have been working hard to improve the status of difference equations. Some of the earliest efforts were in the eighteenth century, when de Moivre, Euler, Lagrange, Laplace and others developed the basic theory of linear difference equations. Later in 1967, the idea of accretive operators in the framework of Banach spaces came to the fore (Sari, 2015). Today, accretive operators are used extensively in finding solutions to problems involving differential equations in Banach spaces with particular emphasis on existence results (Barbu, 1976).

Furthermore, it is instructive to note that the theory of second-order evolution equations of accretive type which later metamorphosed into difference equations of accretive type attracted the interest of many mathematics authors. Some of the pioneer studies in this area are Barbu (1976), Barbu (2010) and Morasanu (1988). Hence, the study of evolution equation is the substratum upon which investigations into difference equation of accretive types have been built and Rouhani *et al.* (2019) acknowledged that only a few studies on (1.1) have been carried out in the framework of Banach spaces, some of which are; Apreutessei (2003), Khtibzadeh (2012), Poffald and Reich (1988).

Ghali and Khabaj (2020), stated that In the early 1960s, Gahler, introduced the 2-Normed spaces, while the concept of 2-Banach spaces was brought to the fore by White in 1969[8,9] (Das, Goswami and Mishra (2017). The study of some properties of accretive operators in connection with 2-Normed spaces was done by Harikrishan and Ravindran (2011), with the focus on contraction mappings and their unique fixed points.

Kir (2013) investigated the accretive operators arising from 2-Banach spaces and derived links between the classes of non-expansive and accretive mappings. Also, Gyegwe and Bassi (2024) studied the non-homogeneous part of (1.1) in the framework of 2-Banach Spaces $(B \ \|.,.\|)$and obtained its existence and uniqueness. In our study, we establish the existence and uniqueness of solutions of (1.1) in the 2-Banach space $(B \ \|.,.\|)$ by using the product space to obtain two m-accretive operators that are also continuous and then expressing them in an expansive mapping form, to obtain two common fixed points. We have also found another fixed-point through the adoption of a 2-Banach contraction mapping.

MATERIALS AND METHODS

In this section, we discuss the terms and notions which are relevant to this study.

Definition 2.1. 2-Normed Space

Let B be real linear space of dimension greater than one and $\|.,.\|$ be a real valued function on $B \times B$ which satisfies the following properties for all $a, b \ c \in B$ and $\alpha \in \mathfrak{R}$.

(i) $\| a, b \| = 0$ if and only if a and b are linearly dependent;
(ii) $\| a, b \| = \| b, a \|$;
(iii) $\| a, \ \alpha b \| = |\alpha| \ \| a, b \|$;
(iv) $\| a, \ b + c \| \leq \| a, b \| \ + \| a, c \|$;

Then the function $\|.,.\|$ is known as a 2-norm on B and the pair $(B, \|.,.\|)$ is described as a linear 2-normed space (Harikrishnan *et al.*, 2021). For example, Let $B = \mathfrak{R}^2$be equipped with the 2-norm $\|.,.\|$. Thus for $a = (a_1, a_2), \ b = (b_1, b_2) \in B = \mathfrak{R}^2$**,** the Euclidean 2-norm $\|.,.\|$. is defined by $\| a, b \| = \| a_1 b_2 - a_2 b_1 \|$. This in a geometrical sense represents the area of the

parallelogram determined by the vectors a and b as the adjacent sides (Eskandani and Rassians, 2016).
Another example is that, suppose P_j denote the set of polynomials of degree $\leq j$, on the interval $[0,1]$. By addition and scholar multiplication process, P_j is a linear vector space over the real field. Also, suppose that $\{a_0, a_1, \dots, a_{2j}\}$ is distinct fixed points $[0,1]$ and define the following 2-norm on P_j : $\| u, v \| = \sum_{k=0}^{2j} |g(a_k)h(a_k)|$, whenever u and v are linearly independent and$\| u, v \| = 0$, if u and v are linearly dependent. Then,$(P_j, \|.,.\|)$is a 2-Banachspace Saha *et al*. (2012).
Remark: Eskandani and Rassians (2016) opines that the basic and fundamental properties of 2-Normed spaces are said to be non-negative and they also satisfy $\| a, b + \alpha a \| = \| a, b \|$, for all $a, b \in B$ and $\alpha \in \mathcal{R}$. Besides, from (iv) above, $\| a, b - c \| = \| a, b \| + \| a, c \|$ or $\| a, b + c \| = a, b \| + \| a, c \|$ [12]. It follows then that from (iv) $|\| a, c \| - \| b, c \|| \leq \| a - b, c \|$. Hence, $a \longmapsto \| a, b \|$ are continuous functions of B into $\mathcal{R}$ for each fixed $b \in B$.
Definition 2.2. Convergent Sequence in 2-Normed Spaces
A sequence (b_j) in a linear 2-normed space B is said to be convergent if there is a $b \in B$ such that $\lim_{j\to\infty} \| b_j - b, c \| = 0$ for all $c \in B$ (Kumar and Pitchaimani, 2019).
Definition 2.3. Cauchy Sequence in 2-Banach Spaces
A linear 2-Normed space $(B, \|.,.\|)$ in which every Cauchy sequence is convergent is called a 2-Banach space (Kir, 2013).
Definition 2.4. Accretive Operator in 2-Banach Spaces
Let$(B, \|.,.\|)$be alinear2-Banach space and a non-linear operator mapping a subset of B. $A: D(A) \subset B$is said to be accretive if for every $a, b, c \in D(A)$ and $\alpha > 0$, then $\| a - b, c \| \leq \| (a - b) + \alpha(Aa - Bb), c \|$.
Additionally, an accretive operator is said to be m-accretive provided that $R(I + \alpha A = B)$, where I is the unity operator in B and R is the range of the map (Kir, 2013).
Definition 2.5. Strong Accretive Operator in 2-Norm
Let $(B, \|.,.\|)$ be a linear 2-norm space, then the mapping $A; B \longrightarrow B$ is said to be strong accretive if for every $D \in B, \| (\alpha - k)(a - b), c \| \leq \| (\alpha - 1)(a - b) + (Aa - Ab), c \|$ for all $a, b \in B$, $\alpha > k$ and $k \in (0,1)$ (Harikrishnan and Ravindran, 2011).
Definition 2.6. Expansive Mapping in 2-Norm
Let $(B, \|.,.\|)$ be a2-Banach space with the 2-norm $\|.,.\|$. A mapping A of B into itself is said to be expansive if there exists a constant $\alpha > 1$ such that $\| Aa - Ab, c \| \geq \alpha \| a - b, c \|$ for all $a, b, \in B$ (Chouhan and Malviya, 2013).
In the following section, we obtain the existence and uniqueness of solution to the homogeneous difference equation (1.1) in 2-Banach spaces by adopting the methods in Poffald and Reich (1986), as well as Apreutesei (2013) using the product space to obtain an expansive mapping. Also, we use the principle of Chouhan and Malviya (2013) to obtain 2-Normed space fixed points from the expansive mapping produced. Finally, we obtain a fixed point from a 2-Banach contraction mapping as done by Gyegwe and Bassi (2024) for the non-homogenous part of (1.1), as stated above.

RESULTS AND DISCUSSION

Let us start with this boundary value problem;

$$\begin{cases} u_{j+1} - (1 + \beta_j)u_j + \beta_j u_{j-1} \in c_j A u_j, \quad 1 \leq j \leq N \\ \qquad u_0 = g\,, \quad u_{N+1} = h \end{cases} \qquad (\mathbf{3.1})$$

Where N is a positive integer, $\{c_j\}_{1 \leq j \leq N}$ and $\beta_j > 0$ are sequences of real numbers. We denote product space containing all N-tuples $u = (u_1, u_2, u_3, \dots, u_N)$ by B^N with $u_j \in B$ for all $1 \leq j \leq N$ given the norm
$\| u, v \| = \sum_{k=1}^{2j} |g(a_k)h(a_k)|$.
Theorem 3.1. Let $(B, \|.,.\|)$ be a 2-Banach space having an m-accretive operator $A: B \times B$. Let the sequence $\{\beta_j\}_{1 \leq j \leq N}$ be non-increasing with $\beta_j > 0$ and $c_j > 0$. Then for each$a, b, c \in B$, the problem (3.1) has aunique solution in B^N.
Proof. We consider the operators A and D defined by $Au = \{(c_1 v_1 \dots, c_N v_N),\ v_j \in A u_j,\ 1 \leq j \leq N\} + (\beta_1 g, 0, \dots, 0, h)$, and denoted by $A \subset B^N \longrightarrow B^N$, where $u = (u_1, \dots, u_N) \in D(A^N)$ and $D: B^N \times B^N$. The operator $Du = ((1 + \beta_1)u_1 - u_2, -\beta_2 u_1 + (1 + \beta_2)u_2 - u_3, \dots, -\beta_{N-1}u_{N-2} + (1 + \beta_{N-1})u_N - u_N, -\beta_N u_{N-1} + (1 + \beta_N)u_N)$.
Hence, the operator $A \subset B^N \times B^N$ is m-accretive and the operator $D: B^N \longrightarrow B^N$ is continuous and everywhere defined and at the same time strongly accretive. Therefore, $A + D$ is m-accretive and continuous. Since A and D are continuous mappings f B into itself, it makes sense to express them as expansive mappings and thus write

$$\| Ag - Dh, c \| \leq c_1 \max\{\| g - h, c \|, \| g - Ag, c \|, \| h - Dh, c \|\} \qquad (\mathbf{3.2})$$

For every $g, h \in B$, $h \in B$, $g \neq h$ where $c_1 > 0$ with A and D having a common fixed point in B. Let the sequence $\{g_j\}$ be defined thus: for $j = 0, 1, 2, 3, \dots$
$g_j = A g_{j+1}$, $g_{j+1} = D g_{j+2}$. We can now plug $g = g_{j+1}$ and $h = g_{j+2}$ in (3.2) to obtain $\| A g_{j+1} - D g_{j+2},\ c \| \geq c_1 \max\{\| g_{j+1} - g_{j+2}, c \|, \| g_{j+1} - A g_{j+1},\ c \|, \| g_{j+2} - D g_{j+2}, c \|\}$

$$= c_1 \max\{\| g_{j+1} - g_{j+2}, c \|, \| g_{j+1} - g_j, c \|, \| g_{j+2} - g_{j+1}, c \|\}$$

$\Rightarrow \| g_j - g_{j+1}, c \| \geq c_1 \max\{\| g_{j+1} - g_{j+2}, c \|, \| g_j - g_{j+1}, c \|\}$.

Case 1
$\| g_j - g_{j+1}, c \| \geq c_1 \| g_{j+1} - g_j, c \|$ This implies that $0 \geq c_1$ which contradicts the assumption that $c_1 \geq 0$.

Case 2

Here, $\| g_{j+1} - g_{j+2}, c \| \leq \frac{1}{c_1} \| g_j - g_{j+1}, c \|$

$\| g_{j+1} - g_{j+2}, c \| \leq \alpha \| g_j - g_{j+1}, c \|$ where $\alpha = \frac{1}{c_1} > 0$

This implies in a general sense that $\| g_j - g_{j+1}, c \| \leq \alpha \| g_{j-1} - g_j, c \|$ for $n = 1, 2, 3, \ldots$ which also implies that

$$\| g_j - g_{j+1}, c \| \leq \alpha^j \| g_0 - g_1, c \| \qquad (3.3)$$

Hence, we can now use (3.3) to prove that $\{g_j\}$ is a Cauchy sequence, meaning that there exists a point $g \in B$ such that

$$\{g_j\} \longrightarrow g \text{ as } j \longrightarrow \infty \qquad (3.4)$$

1.1.1 Existence of fixed points

$g = \lim_{j\to\infty} g_j = \lim_{j\to\infty} A g_{j+1} = A \lim_{j\to\infty} g_{j+1} = Ag \; (as\ j \longrightarrow \infty, \{g_{j+1}\} \longrightarrow g)$. In the same vein,

$g = \lim_{j\to\infty} g_{j+1} = \lim_{j\to\infty} D g_{j+1} = D \lim_{j\to\infty} g_{j+1} = Dg \; (as\ j \longrightarrow \infty, \{g_{j+1}\} \longrightarrow g)$. Thus, we have a common fixed point

$$g = Ag = Dg \qquad (3.5)$$

1.1.2 Existence of uniqueness

We know that for every $(B, \|.,.\|)$, $\lim_{j\to\infty} g_j = g$ and $\lim_{j\to\infty} h_j = h$. Similarly, $\lim_{j\to\infty} Ag_j = Ag$ and $\lim_{j\to\infty} Ah_j = Ah$. If $\lim_{j\to\infty} Ag_j \neq \lim_{j\to\infty} Ah_J$, that forms a contradiction. So, $\lim_{j\to\infty} Ag_J = \lim_{j\to\infty} Ah_J$ which implies that $Ag = Ah \implies g$

Theorem 3.2. Let $(B, \|.,.\|)$be a 2-Banach space that is uniformly and strictly convex. Let the accretive mapping $A: B \longrightarrow B$ form a contraction in the homogeneous second order difference equation of accretive type (1.1), then (1.1) has a unique solution.

Proof. In the first place, we show that (1.1) has a contraction which is the accretive mapping $A: B \longrightarrow B$, for $A^{-1}0 \neq \emptyset$. We define

$$Au_j = u_{j+1} - (1 + \beta_j)u_j + \beta_j u_{j-1} - c_j Au_j \; (3.6)$$

We have to show that for $u, w, z \in B$ and $0 < \alpha < 1$,

$$\| Au - Aw, z \| \leq \alpha \| u - w, z \| \qquad (3.7)$$

From (3.6), we obtain

$$Au_j - Aw_j = (u_{j+1} - w_{j+1}) - (1 + \beta_j)(u_j - w_j) + \beta_j (u_{j-1} - w_{j-1}) + c_j A(u_j - w_j)$$

$\| Au_j - Aw_j, z \| = \| (u_{j+1} - w_{j+1}) - (1 + \beta_j)(u_j - w_j) + \beta_j (u_{j-1} - w_{j-1}) + c_j A(u_j - w_j), z \|$. By triangle inequality, we have

$$\| Au - Aw, z \| \leq \| u - w, z \| + |1 + \beta| \| u - w, z \| + |\beta| \| u - w, z \| + |cA| \| u - w, z \|$$

From(3.8), we see that on the right–hand-side

$2|1 + \beta| + |cA| \| u - w, z \| = \alpha \| u - w, z \|$

$for\ (\alpha \leq 2|1 + \beta| + |cA|)$ which is a contradiction. Therefore

$$\| Au - Aw, z \| \leq \alpha \| u - w, z \| \text{ for all } u, w, z \in B \qquad (3.9)$$

Next we show that from the mapping$A: B \longrightarrow B$, there exists a unique fixed such as $Ag = g$. Let $\alpha \in (0, 1)$ such as $\| Au - Aw, z \| \leq \alpha \| u - w, z \|$ for all $u, w, z \in B$. Also, let u_0 be a point (in fact, an arbitrary one) in B, and let $u_j = Au_{j-1}$ for $j = 1,2,3, \ldots$. We need to prove that $\{u_j\}$ is a Cauchy sequence. First, we can see that for any $j \in \mathbb{N}$,

$\| u_{j+1} - u_j, z \| \leq \alpha \| u_j - u_{j-1}, z \| \leq \alpha^2 \| u_{j-1} - u_{j-2}, z \| \leq \cdots \leq \alpha^j \| u_1 - u_0, z \|$. Hence, $k \in \mathbb{N}$such that$j < k$, we have

$\| u_j - u_k, z \| \leq \| u_j - u_{j-1}, z \| + \| u_{j-1} - u_{j-2}, z \| + \cdots + \| u_{j+1} - u_j, z \|$. Hence, we see that for any $j, k \in \mathbb{N}$ such that $j < k$ we have $\| u_j - u_k, z \| \leq \| u_j - u_{j-1}, z \| + \| u_{j-1} - u_{j-2}, z \| + \| u_{j-2} - u_{j-3}, z + \cdots + \| u_{k+1} - u_k, z \| \leq (\alpha^{j-1} + \alpha^{j-2} + \alpha^{j-3} + \cdots + \alpha^k) \| u_1 - u_0, z \|$

$\leq \frac{\| u_1 - u_0, z \|}{1 - \alpha} \alpha^k \longrightarrow 0$ as$k \longrightarrow \infty$. Similarly, it can be recalled that for the sequence $\{u_j\}$ in a linear 2-normed space, if $\lim_{j,k\to\infty} \| u_j - u_k, z \| = 0$ for all $z \in B$, then $\{u_j\}$ is a Cauchy sequence in B.

Also, for a 2-Banach space$B$, suppose that $g \in B$ such that $u_j \longrightarrow g$ as $j \longrightarrow \infty$, we will show that g is a fixed point such that $Ag = g$. Hence, there is an arbitrary $g \in B$(since B is complete) such that, $u_j \longrightarrow g$ as $j \longrightarrow \infty$. We shall prove that g is a unique fixed point in B. In fact,

$$\| Ag - g, z \| \leq \| Ag - u_j, z \| + \| u_j - g, z \|$$
$$= \| Ag - Au_{j-1}, z \| + \| u_j - g, z \|$$

$\leq \alpha \| g - u_{j-1}, z \| + \| u_j - g, z \| \longrightarrow 0$ as $j \longrightarrow \infty$. Then, we have $\| Ag - g, z \| \longrightarrow 0$. Suppose that we also have $h \in B$ such that $Ah = h$ and $\| Ah - h, z \|$, $h \in B$, then $\| Ah - h, z \| = 0$. Thus, it is reasonable to state that $\| Ag - g, z \| = \| Ah - h, z \|$

Therefore, $(A - 1)g = (A - 1)h \implies g = h$. Hence, the uniqueness of the fixed point $Ag = g$

CONCLUSION

In view of the findings in Theorem 3.1 and Theorem 3.2, we state that the homogeneous second order difference equation of accretive type (1.1) has solutions in 2-Banach spaces, and each solution is unique.

ACKNOWLEDGEMENT

The authors are very grateful to everyone including organization who contributed in every little way to the success of this study.

REFERENCES

Apreutessei, N. C. (2003). Existence and Asymptotic Behaviour for Some Difference Equations Associated with Accretive Operators. In: Vioeri Barbu, *et al. (Eds), Analysis and Optimization of Differential Systems*, IFIP International Federation for Information Processing. DOI: 10.1007/978-387-35690-7_44

Barbu, V. (1976). *Nonlinear Semi groups and Differential Equations in Banach Spaces*. Noordhoff International Publishing, Leiden.

Barbu, V. (2010). *Nonlinear Differential Equations of Monotone Types in Banach Spaces*. Springer Monographs. DOI:10.1007/978-1-4419-5542-5

Chouha, P. and Malviya, N. (2013). Fixed points of expansive type mapping in 2-Banach spaces. *Int. J. of Analysis and Applications*, 3(1), 60-67. http://www.etamaths.com

Das, D., Goswami, N. and Mishra, V. N. (2017). Some fixed point theorems in the projective tensor product of 2-Banach spaces. *Global Journal of Advanced Research on Classical and Modern Geometrics*, 6(1), 20-36.

Eskandani, G. Z. and Rassians, J. M. (2016). Approximation of general $\alpha -$cubic functional equations in 2-Banach spaces. *YKP. Mam.* ⊃ +⊂*ypH*, 68(10), 1027-3190.

Gurdal, M. and Savas, E. (2021). A note on uniquely remotal set in 2-Banach spaces. *Electronic J. of Mathem. Analysis and Application*, 9(2), 151-157. https://maths-frac.org/Journal/EJMAA

Gyegwe, G. T. and Bassi, I. G. (2024). On the existence and uniqueness of solutions to a non-homogeneous second order difference equation of accretive type of 2-Banach spaces. *International Journal of Mathematics and Computer Research* (IJMCR), 12(1), 4051-4054. DOI: 10.47191/jmcr/v 12i2.07

Harikrishnan, P. K. and Ravindran, K. T. (2011). Some properties of accretive operators in linear 2-Normed spaces. *International Mathematical Forum*, 6(59), 2941-2947.

Harikrishnan, P., Guillen, B. L., Agawral, R. P. and Moradi, H. R. (1986). Strong AHD weak convergence in 2-Probailistic normed spaces. *Advances in the Theory of Nonlinear Analysis and its Applications*, 5(4), 454-466. DOI: 10.1031197/atnaa.921900

Kabbaj, R. E. S. (2020). 2-2Banach stability results for radical cubic functional equation related to quadratic mapping. *J. of Linear and Topological Algebra*, 09(1), 35-51. http://jlta.iaucth.ac.ir

Khatibzadeh, H. (2012). Convergence of solutions to a second -order difference inclusion. *Nonlinear Analysis*, 75, 3503-3509. DOI: 10.1016/j.na.2012.01.010

Kir, M. (2013). A new approach to the accretive operators arising from 2-Banach spaces. *General Math Notes*, 16(2), 48-54. http://www.geman.in

Kumar, D. R. and Pitchaimani, M. (2019). Approximation of common fixed point in 2-Banach spaces with applications. *Applied General Topology*, 20(1), 43-55.

Lakshmikanthan, V. and Trigiante, D. (2003). Theory of Difference Equations-Numerical method and Applications, 2nd ed. *Bulletin (New Series) of the American Mathematical Society*, 40(2), 259-262.

Morasanu, G. (1988). *Nonlinear Evolution Equations and Applications*. Editura Academiei Romane (D. Reidei Publishing Company), Bucharest.

Pilakkat, R. and Thirumangalath, S. (2021). Results in linear 2Normed spaces analogous to Baire's theorem and closed graph theorem. *International Journal of Pure and Applied Mathematics*, 74(4), 509-517, 2012.

Poffald, E. I. and Reich, S. (1986). An incomplete cauchy problem. *Journal of Mathematical Analysis and Applications*, 113, 514-543.

Rouhani, B. D. and Khatibzadeh, H. (2012). Existence and asymptotic behaviour of solutions to first and second-order difference equations with periodic forcing. *Journal of Difference Equations and Application*, 18(9), 1593-1606. http://www.tandfonline.com/loi/gdea20

Rouhani, B. D., Jamshidnezhad, P. J. and Saeidi, S. (2019). Zeros of accretive operators and asymptotics of a second-order differense inclusion in Banach spaces. *J. Math. Anal.* 480, 123428. DOI: 10.1016/j.jmaa.2019.123428

Saha, M., Dey, D., Ganguly, A. and Debnath, L. (2012). Asymptotic regularity and fixed point theorems on a 2-2Banach spaces. *Surveys in Math. and its Applic.*, 7, 31-38. http://utgjiu.ro/math/sma

Sari, C. K. (2015). Some properties of m-accretive operators in Normed spaces. *International Conference on Science, Technology and Humanity*, 2477-3328.