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VIBRATION ANALYSIS OF TIMOSHENKO BEAM SUBJECTED TO PARTIALLY DISTRIBUTED MOVING LOAD

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ABSTRACT
Vibration analysis of a Timoshenko beam subjected to a partially distributed load was considered in this paper. The governing equation of fourth-order partial differential equation was reduced to a second-order Ordinary Differential Equation by normalizing the governing equation. The reduced second-order equation was further reduced to a first order ordinary differential system and was solved using fourth-order Runge-Kunta Method. The deflection for various parameters of the beam was considered and was plotted against x. It is observed that the vibration of the beam initially moves in a steady state before deflecting and the deflected amplitude increases as the Shear modulus, distance covered by the load, Mass of the load, Shear coefficient but decreases as the length of the load, the coefficient of the Shear foundation increases.

Keywords: *Vibration, Timoshenko, Beam, Moving load, Distributed partially, Normalize, Shear coefficient, Shear foundation, Amplitude.*

INTRODUCTION

In recent years, all branches of transport have experienced great advances characterized by increasing higher speeds and weight of vehicles (Reference). As a result, structures and media over or in which the vehicles move have been subjected to vibrations and dynamic stresses far larger than ever before. Many scholars have studied vibration of elastic and inelastic structures under the action of moving loads for many years, and effort is still being made to carry out investigation dealing with various aspect of the problem (Cite the scholars). The structures on which these moving loads are usually modelled are by elastic beams, plates or shells. The problem of an elastic beam under the action of the moving loads was considered by Willis (1951). He made the assumption that the mass of the beam is smaller than that of the load and obtained an approximate solution to the problem. Yoshida (1971) studied the vibration of a beam subjected to moving concentrated mass using finite element method. A Simply supported beam subjected to a constant moving force at uniform speed was considered by Krylov (1995) who used the method of expansion of the associated eigenmodes. He assumed the mass of the load to be smaller than that of the beam. Bolotin (1964) carried out a dynamic analysis of the problem involving a concentrated mass traversing a simply supported beam at a constant speed. His approach involves using Galerkin's method. The response of finite simply supported Euler- Bernoulli beam to a unit force moving at a uniform velocity was investigated by Lee (1994). The effects of this moving force on beams with and without an elastic foundation were analyzed. In all the studies discussed above, it was only the force effect of the moving loads that was taken into consideration. The moving load problem involving both the inertia effect as well as the force effects were not considered for several years. This type of dynamical problem was first considered by Kalker (1996), later by Jeffcott (2000) whose iterative method became divergent in some cases. Recently, Esmailzadeh and Gorashi (1995) worked on the vibration analysis of beams traversed by uniform a partially distributed moving masses using analytical-numerical method. They discovered that the inertia effect of the distributed moving mass is of importance in the dynamic behaviour of the structure. The critical speeds of the moving load were also calculated for the mid-span of the beam. The length of the distributed moving mass was also found to affect the dynamic response. The effects of the speed of the moving load, the foundation stiffness and the length of the beam on the response of the beam have been studied and dynamic amplifications of deflection and stress have been evaluated. Based on the Lagrangian approach, Chang (2000) analyzed the vibration of a multi-span non-uniform bridge subjected to a moving vehicle by using modified beam vibration functions as the assumed modes. The structural system can be either steel braces bolted to corner regions of the open bay space in the frame or an infill wall with gaps around the edges to prevent stiffness interaction of the wall with the frame members. Friction dampers are used as sacrificial or non-sacrificial elements. Their utilization as sacrificial elements is a very common attitude in civil engineering environment. In earthquake engineering applications, some of the structural members might be sacrificed in order to prevent the collapse of entire structure. These structural members absorb and dissipate the transmitted energy through plastic deformation in specially detailed regions. Location of the friction damper and stiffness of the braces which are used in order to install dampers are the main factors that affect the design parameters of the damper (Nguyen, 2011) Dahlberg (1999) uses the modal analysis technique to investigate the influence of modal cross-spectral densities on the spectral densities of some responses of simply supported beams. The random response of damped beams was studied by Jacquot (2000). The author presents a method of vibration analysis using the response power spectral density function and mean square response of considered beam structures excited by a second stationary random process. Kukla and Skalmierski (1993) dealt with the random vibration of a clamped-pinned beam. The flux of energy which is emitted by the vibrating beam was investigated. Papadimitriou *et al.,* (2005) provide a methodology for optimal establishment of the number and location of sensors on randomly vibrating structures for the purpose of the response predictions at unmeasured locations in structural systems. The author referees the results of considerations to randomly vibrating beams and plates. Its well known that damping becomes important when the need to have a thorough understanding of the control and mechanical response of vibrating structures arises. An asymptotic analysis of eigen frequencies of uniform beam with both structural and viscous damping coefficient has also been carried out in Hankum and Goong (1991) and Huang (1985). Furthermore, Kenny (1954) took up the problem of investigating the dynamic response of infinite elastic beams on elastic foundation when the beam is under the influence of a dynamic load moving with constant speed. Lie included the effects of viscous damping in the governing differential equation of motion. More recently, Oni (1991) considered the problem of a harmonic time variable concentrated force moving at a uniform velocity over a Unite deep beam. The methods of integral transformations are used. In particular, the Unite Fourier transform is used for the length coordinate and the Laplace transform the time coordinate. Series solution, which converges as obtained for the deflection of simply supported beams. The analysis of the solution was carried out for various speeds of the load. Oni (1991) used the Galerkin method to obtain the response to several moving masses of a non-uniform beam resting on an elastic foundation. The effects of the elastic foundation on the transverse displacement of the non-uniform beam were analyzed for both the moving mass and the associated moving force problems. Awodola (2007) worked on the influence of foundation and axial force on the vibration of a simply supported thin (Bernoulli Euler) beam, resting on a uniform foundation, under the action of a variable magnitude harmonic load moving with variable velocity is investigated in the paper. The governing equation is a fourth-order partial differential equation. For the solution of this problem, in the first instance, the finite Fourier sine transformation is used to reduce the equation to a second order partial differential equation. The reduced equation is then solved using the Laplace transformation. Numerical analysis shows that the transverse deflection of the thin beam, resting on a uniform foundation, under the action of a variable magnitude harmonic load moving with variable velocity decreases as the foundation constant increases. It also shows that as the axial force increases, the transverse deflection of the thin beam decreases. Furthermore, Milormir, Stanisic and Hardin (1969) developed a theory describing the response of a Bernoulli-Euler beam under an arbitrary number of concentrated moving masses. The theory is based on the Fourier technique and shows that, for a simply supported beam, the resonance frequency is lower with no corresponding decrease in maximum amplitude when the inertia is considered.

MATERIALS AND METHODS

Consider an elastic beam of length L, Yong's modulus, E with uniform cross sectional A, breed on Timoshenko beam model assumption, the effect of shear deformation and rotary inesta are not negligible, and therefore they are considered in this research.

The vibration of Timoshenko beam subjected to a partially distributed moving load has the governing equation of the form

$$EI\frac{\partial^4 w(x,t)}{\partial t^x} + \rho A\frac{\partial^2 w(x,t)}{\partial t^2} - \rho A\sigma_G^2\frac{\partial^4 w(x,t)}{\partial^2 x\partial t^2}\left(1+\frac{E}{KG}\right) + \frac{\rho^2 A\sigma_G^2}{KG}\frac{\partial^4 w(x,t)}{\partial t^4} = f(x,t) \qquad \text{........... 1}$$

with

$$f(x,t) = \frac{1}{\epsilon}\left[-Mg - M\frac{d^2 w}{dx^2}\right]\left[H\left(x-\xi+\frac{\epsilon}{2}\right) - H\left(x-\xi-\frac{\epsilon}{2}\right)\right] \qquad \text{..2}$$

where

x = spatial coordinate

t = time

$\frac{\partial^n}{\partial x^n}$ = nth partial derivative with respect to x

w(x,t) = deflection of the beam

E = Young's modulus

I = moment of inertia of the beam's cross section about the neutral axis

A = Area

ρ = Density

K = Shear Coefficient

G = Shear Modulus

σG = Modulus of the Shear foundation

The differential operator $\frac{d^2 w(x,t)}{dx^2}$ is defined as

$$\frac{d^2 w(x,t)}{dx^2} = \frac{\partial^2 w(x,t)}{\partial t^2} + 2V\frac{\partial^2 w(x,t)}{\partial x\partial t} + V^2\frac{\partial^2 w(x,t)}{\partial x^2} \qquad \text{...3}$$

H(x) is the heavy-side function such that

$$H\left(x-\xi+\frac{\epsilon}{2}\right) = H\left(x-\left(\xi-\frac{\epsilon}{2}\right)\right) = \begin{cases} 0 & x < \xi - \frac{\epsilon}{2} \\ 1 & x > \xi - \frac{\epsilon}{2} \end{cases}$$

$$H\left(x-\xi-\frac{\epsilon}{2}\right) = H\left(x-\left(\xi+\frac{\epsilon}{2}\right)\right) = \begin{cases} 0 & x < \xi + \frac{\epsilon}{2} \\ 1 & x > \xi + \frac{\epsilon}{2} \end{cases} \qquad \text{,..4}$$

Hence, the governing equation becomes

$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+\rho A\frac{\partial^2 w(x,t)}{\partial t^2}-\rho A\sigma_G^2\frac{\partial^4 w(x,t)}{\partial^2 x\partial t^2}\left(1+\frac{E}{KG}\right)+\frac{\rho^2 A\sigma_G^2}{KG}\frac{\partial^4 w(x,t)}{\partial t^4}$$
$$=\frac{1}{\epsilon}\left[-Mg-\left(M\frac{\partial^2 w(x,t)}{\partial t^2}+2MV\frac{\partial^2 w(x,t)}{\partial x\partial t}+MV^2\frac{\partial^2 w(x,t)}{\partial x^2}\right)\right] \quad ....5$$
$$\left[H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right]$$

With the boundary conditions

$w(0,t) = 0 = w(l,t)$ 6

$$\frac{\partial^2 w(0,t)}{\partial x^2}=0=\frac{\partial^2 w(l,t)}{\partial x^2} \quad ....7$$

Without loss of generality, one can consider the initial conditions of the form

$$w(x,0)=\frac{\partial w(x,0)}{\partial t}=\frac{\partial^2 w(x,0)}{\partial t^2}=\frac{\partial^3 w(x,0)}{\partial t^3} \quad ....8$$

Method of Solution

Assume a solution such that the transverse vibration of the beam may be expressed in the following series form

$$W(x,t)=\sum_{i=1}^{\infty}X_i(x)\lambda_i(t) \quad ,...9$$

Substituting (9) into (5), we have

$$EI\sum_{i=1}^{\infty}X_i^{iv}(x)\lambda_i(t)+\rho A\sum_{i=1}^{\infty}X_i(x)\lambda_i''(t)-\rho A\sigma_G^2\sum_{i=1}^{\infty}X_i''(x)\lambda_i''(t)\left(1+\frac{E}{KG}\right)$$
$$+\frac{\rho^2 A\sigma_G^2}{KG}\sum_{i=1}^{\infty}X_i(x)\lambda_i^{iv}(t)=\left[-\frac{Mg}{\epsilon}-\frac{M}{\epsilon}\sum_{i=1}^{\infty}\lambda_i''(t)X_i(x)-\frac{2MV}{\epsilon}\sum_{i=1}^{\infty}\lambda_i'(t)X_i'(x)\right. \quad ....10$$
$$\left.-\frac{V^2M}{\epsilon}\sum_{i=1}^{\infty}\lambda_i(t)X_i''(x)\right]\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}$$

Furthermore, the forcing term f(x,t) defined in equation (2) can also be expressed as

$$f(x,t)=\sum_{i=1}^{\infty}\lambda_{fi}(t)X_i(x) \quad ....11$$

Substituting (11) into (2), we have

$$\sum_{i=1}^{\infty}\lambda_{fi}(t)X_i(x)=-\frac{Mg}{\epsilon}\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}$$
$$-\frac{M}{\epsilon}\sum_{i=1}^{\infty}\lambda_i''(t)X_i(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}$$
$$-\frac{2MV}{\epsilon}\sum_{i=1}^{\infty}\lambda_i'(t)X_i'(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}$$
$$-\frac{V^2M}{\epsilon}\sum_{i=1}^{\infty}\lambda_i(t)X_i''(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\} \quad ....12$$

To normalize equation (12), we multiply all through by *Xj(x)* to obtain

$$\sum_{i=1}^{\infty}\lambda_{fi}(t)X_i(x)X_j(x)=-\frac{Mg}{\epsilon}X_j(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}$$
$$-\frac{M}{\epsilon}\sum_{i=1}^{\infty}\lambda_i''(t)X_i(x)X_j(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}$$
$$-\frac{2MV}{\epsilon}\sum_{i=1}^{\infty}\lambda_i'(t)X_i'(x)X_j(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}$$
$$-\frac{V^2M}{\epsilon}\sum_{i=1}^{\infty}\lambda_i(t)X_i''(x)X_j(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}$$

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Integrating both sides of (13) with respect to x along the length L of the beam, we have

$$\sum_{i=1}^{\infty}\lambda_{fi}(t)\int_0^L X_i(x)X_j(x)\,dx=-\frac{Mg}{\epsilon}\int_0^L X_j(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx$$
$$-\frac{M}{\epsilon}\sum_{i=1}^{\infty}\lambda_i''(t)\int_0^L X_i(x)X_j(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx$$
$$-\frac{2MV}{\epsilon}\sum_{i=1}^{\infty}\lambda_i'(t)\int_0^L X_i'(x)X_j(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx$$
$$-\frac{V^2M}{\epsilon}\sum_{i=1}^{\infty}\lambda_i(t)\int_0^L X_i''(x)X_j(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx$$

.........14

From (14), we assume the following

$$0=-\frac{Mg}{\epsilon}\int_0^L\int_0^L X_j(x)\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx$$

....15

Integrating by part using

$$H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\int_0^L X_j(x)\,dx$$
$$-\int_0^L\int_0^L X_j(x)\left[H'\left(x-\xi+\frac{\epsilon}{2}\right)-H'\left(x-\xi-\frac{\epsilon}{2}\right)\right]dxdx$$

....16

Since H (x) =∫δ(x)

F (x) = H (x) where H'(x) = δ (x) 17

such that we get,

Let $A_{11}=-\frac{Mg}{\epsilon}\int_0^L X_j\left(\xi+\frac{\epsilon}{2}\right)d\epsilon-\int_0^L X_j\left(\xi-\frac{\epsilon}{2}\right)\epsilon\delta(x)$ 18

Furthermore, expanding using Taylor series, we obtain,

$$X_j\left(\xi+\frac{\epsilon}{2}\right)=X_j(\xi)+\frac{\frac{\epsilon}{2})}{1!}X_j'(\xi)+\frac{\frac{\epsilon}{2})^2}{2!}X_j''(\xi)+\frac{\frac{\epsilon}{2})^3}{3!}X_j'''(\xi)+\cdots$$

....................................19

Also,

$$X_j\left(\xi-\frac{\epsilon}{2}\right)=X_j(\xi)-\frac{\frac{\epsilon}{2})}{1!}X_j'(\xi)+\frac{\frac{\epsilon}{2})^2}{2!}X_j''(\xi)-\frac{\frac{\epsilon}{2})^3}{3!}X_j'''(\xi)+\cdots$$

....20

By substituting (19)−(20) into (18), we have

$$A_{11}=-\frac{Mg}{\epsilon}\int_0^L\int_0^L\left[X_i(\xi)+\frac{\frac{\epsilon}{2})}{1!}X_j'(\xi)+\frac{\frac{\epsilon}{2})^2}{2!}X_j''(\xi)+\frac{\frac{\epsilon}{2})^3}{3!}X_j'''(\xi)-X_j(\xi)\right.$$
$$\left.+\frac{\frac{\epsilon}{2})}{1!}X_j'(\xi)-\frac{\frac{\epsilon}{2})^2}{2!}X_j''(\xi)+\frac{\frac{\epsilon}{2})^3}{3!}X_j'''(\xi)\right]$$

....21

We get,

$$\epsilon X_j'(\xi)+\frac{\epsilon^3}{24}X_j'''(\xi)$$

....22

Substituting (22) into (16) and having satisfy the condition (4), we have,

$$0=-Mg\left[X_j\left(\xi\right)+\frac{\epsilon^2}{24}X_j''\left(\xi\right)\right] \quad ......23$$

Similar arguments is applicable to second, third and fourth definite integral in (14), hence, evaluating the integrals using Taylor's series expansion and applying orthogonality properties of the characteristics function λfi(t) the left hand side of (14), we finally obtain

$$\begin{aligned}\lambda_{fi}\left(t\right)=&-Mg\left[X_j\left(\xi\right)+\frac{\epsilon^2}{24}X_j''\left(\xi\right)\right]\\&-M\sum_{j=1}^{\infty}\lambda''{}_i(t)\left[X_j(\xi)X_i(\xi)+\frac{\epsilon^2}{24}\ \ X_j\left(\xi\right)X_i''\left(\xi\right)+2X_j'\left(\xi\right)X_i'\left(\xi\right)+X_j''\left(\xi\right)X_i\left(\xi\right)\right)\right]\\&-2MV\sum_{j=1}^{\infty}\lambda_i'\left(t\right)\left[X_j\left(\xi\right)X_i'\left(\xi\right)+\frac{\epsilon^2}{24}\ \ X_j\left(\xi\right)X_i'''\left(\xi\right)+2X_j'\left(\xi\right)X_i''\left(\xi\right)+X_j''\left(\xi\right)X_i'\left(\xi\right)\right)\right]\\&-V^2M\sum_{j=1}^{\infty}\lambda_i\left(t\right)\left[X_j\left(\xi\right)X_i''\left(\xi\right)+\frac{\epsilon^2}{24}\ \ X_j\left(\xi\right)X_i^{iv}\left(\xi\right)+2X_j'\left(\xi\right)X_i'''\left(\xi\right)+X_j''\left(\xi\right)X_i''\left(\xi\right)\right)\right]\end{aligned} \quad (3.24) \quad ....24$$

Furthermore, from (10) we have that

$$\begin{aligned}&EI\sum_{i=1}^{\infty}X_i^{iv}\left(x\right)\lambda_i\left(t\right)+\rho A\sum_{i=1}^{\infty}X_i\left(x\right)\lambda_i''\left(t\right)-\rho A\sigma_G^2\sum_{i=1}^{\infty}X_i''\left(x\right)\lambda_i''\left(t\right)\left(1+\frac{E}{KG}\right)\\&+\frac{\rho^2A\sigma_G^2}{KG}\sum_{i=1}^{\infty}X_i\left(x\right)\lambda_j^{iv}\left(t\right)=\sum_{i=1}^{\infty}\lambda_{fi}\left(t\right)X_i\left(x\right.\end{aligned} \quad ....25$$

substituting (24) into (25) becomes

$$\begin{aligned}&EI\sum_{i=1}^{\infty}X_i^{iv}\left(x\right)\lambda_i\left(t\right)+\rho A\sum_{i=1}^{\infty}X_i\left(x\right)\lambda_i''\left(t\right)-\rho A\sigma_G^2\sum_{i=1}^{\infty}X_i''\left(x\right)\lambda_i''\left(t\right)\left(1+\frac{E}{KG}\right)\\&+\frac{\rho^2A\sigma_G^2}{KG}\sum_{i=1}^{\infty}X_i\left(x\right)\lambda_j^{iv}\left(t\right)=-Mg\left[X_j\left(\xi\right)+\frac{\epsilon^2}{24}X_j''\left(\xi\right)\right]\\&-M\sum_{i=1}^{\infty}\lambda''{}_i(t)\left[X_j(\xi)X_i(\xi)+\frac{\epsilon^2}{24}\ \ X_j\left(\xi\right)X_i''\left(\xi\right)+2X_j'\left(\xi\right)X_i'\left(\xi\right)+X_j''\left(\xi\right)X_i\left(\xi\right)\right)\right]\\&-2MV\sum_{i=1}^{\infty}\lambda_i'\left(t\right)\left[X_j\left(\xi\right)X_i'\left(\xi\right)+\frac{\epsilon^2}{24}\ \ X_j\left(\xi\right)X_i'''\left(\xi\right)+2X_j'\left(\xi\right)X_i''\left(\xi\right)+X_j''\left(\xi\right)X_i'\left(\xi\right)\right)\right]\\&-V^2M\sum_{i=1}^{\infty}\lambda_i\left(t\right)\left[X_j\left(\xi\right)X_i''\left(\xi\right)+\frac{\epsilon^2}{24}\ \ X_j\left(\xi\right)X_i^{iv}\left(\xi\right)+2X_j'\left(\xi\right)X_i'''\left(\xi\right)+X_j''\left(\xi\right)X_i''\left(\xi\right)\right)\right]\end{aligned} \quad (3.26) \quad ....26$$

so that (26) becomes

$$EI\sum_{i=1}^{\infty}X_i^{iv}(x)\lambda_i(t)+\rho A\sum_{i=1}^{\infty}X_i(x)\lambda_i''(t)-\rho A\sigma_G^2\sum_{i=1}^{\infty}X_i''(x)\left(1+\frac{E}{KG}\right)\lambda_j''(t)$$
$$+\frac{\rho^2A\sigma_G^2}{KG}\sum_{i=1}^{\infty}X_i(x)\lambda_i^{iv}(t)+Mg\left[X_j(\xi)+\frac{\epsilon^2}{24}X_j''(\xi)\right]$$
$$+M\sum_{i=1}^{\infty}\lambda_i''(t)\left[X_j(\xi)X_i(\xi)+\frac{\epsilon^2}{24}\ X_j(\xi)X_i''(\xi)+2X_j'(\xi)X_i'(\xi)+X_j''(\xi)X_i(\xi))\right] \qquad (3.27)$$
$$+2MV\sum_{i=1}^{\infty}\lambda_i'(t)\left[X_j(\xi)X_i'(\xi)+\frac{\epsilon^2}{24}\ X_j(\xi)X_i'''(\xi)+2X_j'(\xi)X_i''(\xi)+X_j''(\xi)X_i'(\xi))\right]$$
$$+V^2M\sum_{i=1}^{\infty}\lambda_i(t)\left[X_j(\xi)X_i''(\xi)+\frac{\epsilon^2}{24}\ X_j(\xi)X_i^{iv}(\xi)+2X_j'(\xi)X_i'''(\xi)+X_j''(\xi)X_i''(\xi))\right]=0$$

$$\sum_{i=1}^{\infty}X_i^{iv}\left[EI\lambda_i+\rho A\lambda_i''(t)-\rho A\sigma_G^2\left(1+\frac{E}{KG}\right)\lambda_i''(t)+\frac{\rho^2A\sigma_G^2}{KG}\lambda_i^{iv}(t)\right]$$
$$+\ Mg\left[X_j(\xi)+\frac{\epsilon^2}{24}X_j''(\xi)\right]$$
$$+\ M\sum_{i=1}^{\infty}\lambda''{}_i(t)\left[X_j(\xi)X_i(\xi)+\frac{\epsilon^2}{24}\ X_j(\xi)X_i''(\xi)+2X_j'(\xi)X_i'(\xi)+X_j''(\xi)X_i(\xi))\right]$$
$$+\ 2MV\sum_{i=1}^{\infty}\lambda_i'(t)\left[X_j(\xi)X_i'(\xi)+\frac{\epsilon^2}{24}\ X_j(\xi)X_i'''(\xi)+2X_j'(\xi)X_i''(\xi)+X_j''(\xi)X_i'(\xi))\right]$$
$$+\ V^2M\sum_{i=1}^{\infty}\lambda_i(t)\left[X_j(\xi)X_i''(\xi)+\frac{\epsilon^2}{24}\ X_j(\xi)X_i^{iv}(\xi)+2X_j'(\xi)X_i'''(\xi)+X_j''(\xi)X_i''(\xi))\right]=0$$

(3.28) 28

Equation (28) above must be satisfied for arbitrary Xi(x)

$$\sum_{i=1}^{\infty}X_i^{iv}\left[EI\lambda_i+\rho A\lambda_i''(t)-\rho A\sigma_G^2\left(1+\frac{E}{KG}\right)\lambda_i''(t)+\frac{\rho^2A\sigma_G^2}{KG}\lambda_i^{iv}(t)\right]$$
$$=-Mg\left[X_j(\xi)+\frac{\epsilon^2}{24}X_j''(\xi)\right]$$
$$-M\sum_{i=1}^{\infty}\lambda''{}_i(t)\left[X_j(\xi)X_i(\xi)+\frac{\epsilon^2}{24}\ X_j(\xi)X_i''(\xi)+2X_j'(\xi)X_i'(\xi)+X_j''(\xi)X_i(\xi))\right] \qquad (3.29)$$
$$-2MV\sum_{i=1}^{\infty}\lambda_i'(t)\left[X_j(\xi)X_i'(\xi)+\frac{\epsilon^2}{24}\ X_j(\xi)X_i'''(\xi)+2X_j'(\xi)X_i''(\xi)+X_j''(\xi)X_i'(\xi))\right]$$
$$-V^2M\sum_{i=1}^{\infty}\lambda_i(t)\left[X_j(\xi)X_i''(\xi)+\frac{\epsilon^2}{24}\ X_j(\xi)X_i^{iv}(\xi)+2X_j'(\xi)X_i'''(\xi)+X_j''(\xi)X_i''(\xi))\right]=0$$

For the boundary conditions given under the governing equation

$$X_i(x)=\sin\left[\frac{i\pi x}{L}\right]$$ 30

To obtain a set of exact governing differential equation for the simply supported beam under consideration, we substitute (30) into (14) to obtain

$$\sum_{i=1}^{\infty}\lambda_{fi}(t)\int_0^L \sin\left[\frac{i\pi}{L}x\right]\sin\left[\frac{j\pi}{L}x\right]dx$$
$$= -M\frac{g}{\epsilon}\int_0^L \sin\frac{i\pi}{L}x\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx$$
$$-2M\frac{V}{\epsilon L}\sum_{i=1}^{\infty}\lambda_i(t)\int_0^L \sin\frac{i\pi}{L}x\sin\frac{i\pi}{L}x\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx$$
$$-4MV\frac{i\pi}{\epsilon L^2}\sum_{i=1}^{\infty}\lambda_i'(t)\int_0^L \sin\frac{i\pi}{L}x\cos\frac{i\pi}{L}x\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx$$
$$-2MV^2\frac{i^2\pi^2}{\epsilon L^3}\sum_{i=1}^{\infty}\lambda_i(t)\int_0^L \sin\frac{i\pi}{L}x\sin\frac{i\pi}{L}x\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx$$
............31

Evaluating the above integrals, we have

$$Q_1 = \int_0^L \sin\frac{j\pi}{L}x\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx = 2\sin\left[\frac{j\pi}{L}\xi\right]\sin\left[\frac{i\pi}{2L}\epsilon\right]$$

$$\begin{aligned} Q_2 &= \int_0^L \sin\frac{i\pi}{L}x\sin\frac{j\pi}{L}x\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx \\ &= \frac{1}{i-j}\cos\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j)-\frac{1}{i+j}\cos\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j \end{aligned}$$
........................32

$$\begin{aligned} Q_3 &= \int_0^L \sin\frac{i\pi}{L}x\cos\frac{j\pi}{L}x\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx \\ &= \frac{1}{i+j}\sin\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j)-\frac{1}{i-j}\sin\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j \end{aligned}$$

$$\begin{aligned} Q_4 &= \int_0^L \sin\frac{i\pi}{L}x\sin\frac{j\pi}{L}x\left\{H\left(x-\xi+\frac{\epsilon}{2}\right)-H\left(x-\xi-\frac{\epsilon}{2}\right)\right\}dx \\ &= \frac{1}{i-j}\cos\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j)-\frac{1}{i+j}\cos\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j \end{aligned}$$

$$Q_5 = \int_0^L \sin\left[\frac{i\pi}{L}x\right]\sin\left[\frac{j\pi}{2L}x\right]dx = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases}$$

By substituting equations (32)−(36) into (31), we have

$$\lambda_{fi}(t) = -\frac{Mg}{i\pi\epsilon}\sin\left[\frac{j\pi}{L}\xi\right]\sin\left[\frac{i\pi}{L}\epsilon\right]$$
$$-2\frac{M}{\epsilon L}\sum_{i=1}^{\infty}\lambda_i''(t)\left\{\frac{1}{i-j}\cos\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j)-\frac{1}{i+j}\cos\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j)\right\}$$
$$-4MV\frac{i\pi}{\epsilon L^2}\sum_{i=1}^{\infty}\lambda_i'(t)\left\{\frac{1}{i+j}\sin\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j)-\frac{1}{i-j}\sin\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j)\right\}(3.37)$$
$$-2MV^2\frac{i^2\pi^2}{\epsilon L^3}\sum_{i=1}^{\infty}\lambda_i(t)\left\{\frac{1}{i-j}\cos\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j)-\frac{1}{i+j}\cos\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j)\right\}$$

$i\neq j, i=1,2,3,\cdots$ ....

By replacing the right hand side of (29) with the right hand side of (37), we finally obtain

$$EI\frac{\pi^4}{L^4}i^4\lambda_i(t)+\rho A\lambda_i''(t)-\rho A\sigma_G^2\frac{\pi^2}{L^2}\left(1+\frac{E}{KG}\right)i^2\lambda_i''(t)+\frac{\rho^2 A\sigma_G^2}{KG}\lambda_i^{iv}(t)$$
$$=-M\frac{g}{i\pi\epsilon}\sin\left[\frac{i\pi}{L}\xi\right]\sin\left[\frac{j\pi}{2L}\epsilon\right]$$
$$-2\frac{M}{\epsilon L}\sum_{i=1}^{\infty}\lambda_i''(t)\left\{\frac{1}{i-j}\cos\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j)-\frac{1}{i+j}\cos\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j)\right\} \quad (3.38)$$
$$-4MV\frac{i\pi}{\epsilon L^2}\sum_{i=1}^{\infty}\lambda_i'(t)\left\{\frac{1}{i+j}\sin\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j)-\frac{1}{i-j}\sin\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j)\right\}$$
$$-2MV^2\frac{i^2\pi^2}{\epsilon L^3}\sum_{i=1}^{\infty}\lambda_i(t)\left\{\frac{1}{i-j}\cos\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j)-\frac{1}{i+j}\cos\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j)\right\}$$
$$i\neq j, i=1,2,3,\cdots$$
....

$$EI\frac{\pi^4}{L^4}i^4\lambda_i(t)+\rho A\lambda_i''(t)-\rho A\sigma_G^2\frac{\pi^2}{L^2}\left(1+\frac{E}{KG}\right)i^2\lambda_i''(t)+\frac{\rho^2 A\sigma_G^2}{KG}\lambda_i^{iv}(t)=-M\frac{g}{i\pi\epsilon}\sin\left[\frac{i\pi}{L}\xi\right]\sin\left[\frac{j\pi}{2L}\epsilon\right]$$
$$-\frac{2M}{\epsilon L}\sum_{i=1}^{\infty}\lambda_i''(t)\left\{\frac{1}{i-j}\cos\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j)\right\}+\frac{2M}{\epsilon L}\sum_{i=1}^{\infty}\lambda_i''(t)\left\{\frac{1}{i+j}\cos\frac{\pi}{L}\xi(i+j)\right.$$
$$\left.\sin\frac{\pi}{2L}\epsilon(i+j)\right\}-4MV\frac{i\pi}{\epsilon L^2}\sum_{i=1}^{\infty}\lambda_i'(t)\left\{\frac{1}{i+j}\sin\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j)\right\}$$
$$+4MV\frac{i\pi}{\epsilon L^2}\sum_{i=1}^{\infty}\lambda_i'(t)\left\{\frac{1}{i-j}\sin\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j)\right\}-2MV^2\frac{i^2\pi^2}{\epsilon L^3}\sum_{i=1}^{\infty}\lambda_i(t)$$
$$\left\{\frac{1}{i-j}\cos\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j)\right\}+2MV^2\frac{i^2\pi^2}{\epsilon L^3}\sum_{i=1}^{\infty}\lambda_i(t)\left\{\frac{1}{i+j}\cos\frac{\pi}{L}\xi(i+j)\right.$$
$$\left.\sin\frac{\pi}{2L}\epsilon(i+j)\right\}\quad i=1,2,3,\cdots,i\neq j$$
....

$$EI\frac{\pi^4}{L^4}i^4\lambda_i(t)+\rho A\lambda_i''(t)-\rho A\sigma_G^2\frac{\pi^2}{L^2}\left(1+\frac{E}{KG}\right)i^2\lambda_i''(t)+\frac{\rho^2 A\sigma_G^2}{KG}\lambda_i^{iv}(t)$$
$$=-M\frac{g}{i\pi\epsilon}\sin\left[\frac{i\pi}{L}\xi\right]\sin\left[\frac{j\pi}{2L}\epsilon\right]-\frac{2M}{\epsilon L}\sum_{i=1}^{\infty}\lambda_i''(t)$$
$$\left\{\frac{1}{i-j}\cos\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j)\right\}+\frac{2M}{\epsilon L}\sum_{i=1}^{\infty}\lambda_i''(t)\left\{\frac{1}{i+j}\cos\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j)\right\}$$
$$-4MV\frac{i\pi}{\epsilon L^2}\sum_{i=1}^{\infty}\lambda_i'(t)\left\{\frac{1}{i+j}\sin\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j)\right\}+4MV\frac{i\pi}{\epsilon L^2}\sum_{i=1}^{\infty}\lambda_i'(t)\left\{\frac{1}{i-j}\right.$$
$$\left.\sin\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j)\right\}-2MV^2\frac{i^2\pi^2}{\epsilon L^3}\sum_{i=1}^{\infty}\lambda_i(t)\left\{\frac{1}{i-j}\cos\frac{\pi}{L}\xi(i-j)\right.$$
$$\left.\sin\frac{\pi}{2L}\epsilon(i-j)\right\}+2MV^2\frac{i^2\pi^2}{\epsilon L^3}\sum_{i=1}^{\infty}\lambda_i(t)\left\{\frac{1}{i+j}\cos\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j)\right\}$$
....

$$\frac{\rho^2 A\sigma_G^2}{KG}\lambda_i^{iv}(t) + \left\{\rho A - \rho A\sigma_G^2\frac{\pi^2}{L^2}\left(1+\frac{E}{KG}\right)i^2\right.$$
$$+\frac{2M}{\epsilon L}\left[\frac{1}{i-j}\cos\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j) - \frac{1}{i+j}\cos\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j)\right]\Bigg\}\lambda_i''(t)$$
$$+\left\{2MV\frac{i\pi}{\epsilon L^2}h\left[\frac{1}{i+j}\sin\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j) - \frac{1}{i-j}\sin\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j)\right]\right\}\lambda_i'(t)$$
$$\left\{EI\frac{\pi^4}{L^4}i^4 - MV^2\frac{i^2\pi^2}{\epsilon L^2}\left[\frac{1}{i-j}\cos\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j)\right.\right.$$
$$\left.\left.-\frac{1}{i+j}\cos\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j)\right]\right\}\lambda_i(t) = -2M\frac{gL}{i\pi\epsilon}\sin\left[\frac{i\pi}{L}\xi\right]\sin\left[\frac{j\pi}{2L}\epsilon\right] \quad ....$$

Hence, we have

$$\lambda_i^{iv}(t) + E\lambda_i''(t) + F\lambda_i'(t) + G\lambda_i(t) = H \quad ....42$$

where

$$E = \frac{B}{I},\ F = \frac{C}{I},\ G = \frac{D}{I},\ H = \frac{K}{I}$$

$$I = \frac{\rho^2 A\sigma_G^2}{KG}$$

$$B = \rho A - \rho A\sigma_G^2\left(1+\frac{E}{KG}\right)\frac{\pi^2}{L^2}i^2 + \frac{2M}{\epsilon L}\left[\frac{1}{i-j}\cos\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j)\right.$$
$$\left.-\frac{1}{i+j}\cos\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j)\right]$$

$$C = 2MV\frac{i\pi}{\epsilon L^2}h\left[\frac{1}{i+j}\sin\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j) - \frac{1}{i-j}\sin\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j)\right]$$

$$D = EI\frac{\pi^4}{L^4}i^4 - MV^2\frac{i^2\pi^2}{\epsilon L^2}\left[\frac{1}{i-j}\cos\frac{\pi}{L}\xi(i-j)\sin\frac{\pi}{2L}\epsilon(i-j) - \frac{1}{i+j}\cos\frac{\pi}{L}\xi(i+j)\sin\frac{\pi}{2L}\epsilon(i+j)\right]$$

$$K = -2M\frac{gL}{i\pi\epsilon}\sin\left[\frac{i\pi}{L}\xi\right]\sin\left[\frac{j\pi}{2L}\epsilon\right]$$

In order to solve (42), we reduce it to a first order system of equations and then use fourth-order Runge-Kunta Method to solve the system. i.e.

$$\lambda = \lambda 1$$
$$\lambda_1' = \lambda_2$$
$$\lambda_2' = \lambda_3 \quad ....(43)$$
$$\lambda_3' = \lambda_4$$
$$\lambda_4' = H - E\lambda_4 - F\lambda_2 - G\lambda_1$$

Representing (43) in form of AX + B = C

$$\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix}' = \begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -G & 0 & -F & -E\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix} + \begin{pmatrix}0\\ 0\\ 0\\ H\end{pmatrix} \quad ....44$$

To solve equation (44), the following fourth-order Runge-Kunta method is used

$$\lambda_{i+1} = \lambda_i + h\left[\frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6}\right] \quad ....45$$

with

$$k_1 = f(t_i, \lambda_i)$$

$$k_2 = f(t_i + \frac{h}{2}, \lambda_i + \frac{h}{2}k_1)$$

$$k_3 = f(t_i + \frac{h}{2}, \lambda_i + \frac{h}{2}k_2) \quad ....46$$

$$k_4 = f(t_i + h, \lambda_i + hk_3)$$

Equation (44) is solved numerically with the aid of Matlab which is used to carry out analysis of the next session.

Therefore, the deflection of Timoshenko beam is

$$w(x,t) = \lambda_{i+1}(t)\sin\frac{i\pi}{L}x \quad ....47$$

where, λ_{i+1} is given in equation (45)

RESULTS AND DISCUSSION

Beam dimension and specification:

The beam was made of steel $E=2.10\times10^{11}$N, Length(L)=10m, Density of the mass (ρ) = 1.64×10^8kg/m^3, Surface area of the beam cross section $A=6\times10^{-6}$m^2, Shear coefficient K=0.5, 1.0, 1.5, Shear modulus G=0.2, 0.4, 0.8, Distance covered by the load ξ=0.1, 0.2, 0.3, Load's length , Modulus of shear foundation σG = 0.1, 0.2, 0.3, Rigidity of the Beam EI = 1.74×10^{-5}m^4

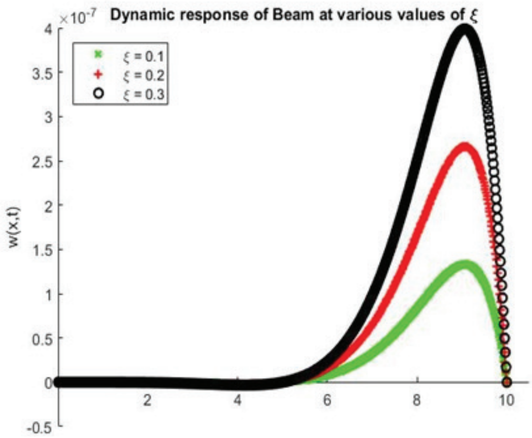

Figure 1: The effect of ξ on the deflection of beam

Figure (1) shows the dynamic response of the beam at various values of the distance covered by the load. It is observed that it first move in a steady state before deflecting and the deflected amplitude increases as the distance covered by the load increases.

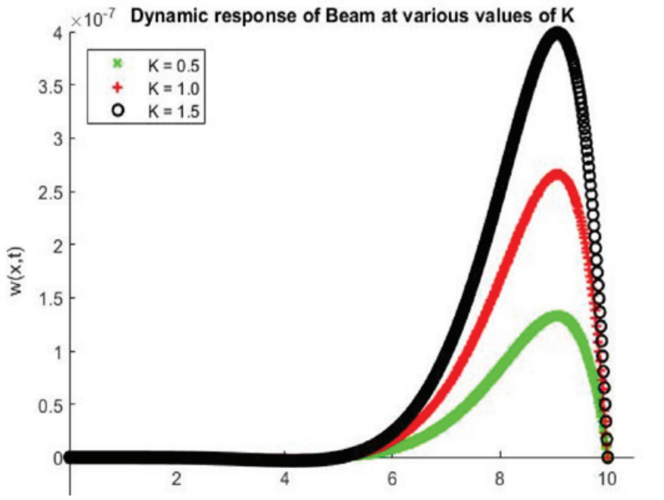

Figure 2: The effect of K on the deflection of beam

Figure 2 shows the dynamic response of the beam at various values of shear coefficient. It is observed that the vibration of the beam initially moves in a steady state before deflecting and the deflected amplitude increases as the shear coefficient increases

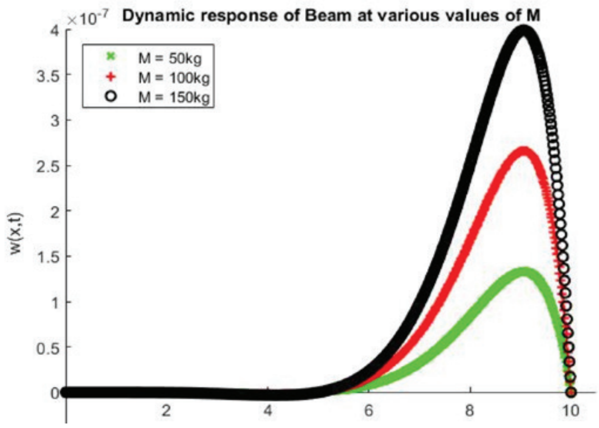

Figure 3: The effect of M on the deflection of beam

Figure 3 displays the dynamic response of the beam at various values of the mass of the load. It is observed that the vibration of the beam initially moves in a steady state before deflecting and the deflected amplitude increases as the mass of the load increases.

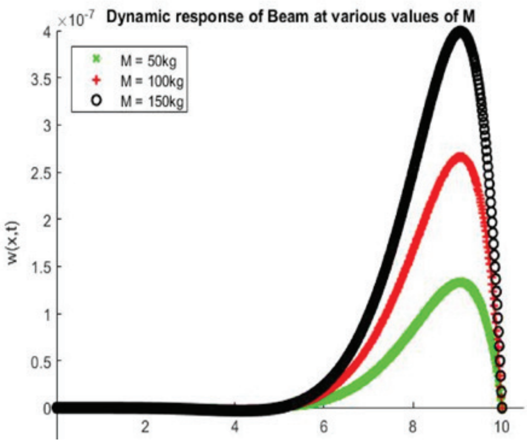

Figure 4: The effect of M on the deflection of beam

Figure 4 shows the dynamic response of the beam at various values of the coefficient of modulus of shear foundation. It is observed that the vibration of the beam initially moves in a steady state before deflecting and the deflected amplitude decreases as the modulus of shear foundation increases.

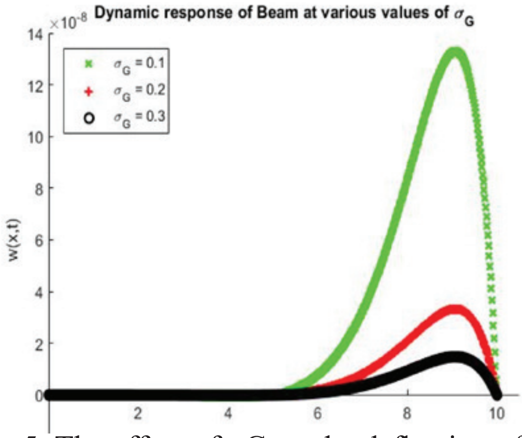

Figure 5: The effect of σG on the deflection of beam

Figure 5 shows the dynamic response of the beam at various values of Shear modulus. It is observed that the vibration of the beam initially moves in a steady state before deflecting and the deflected amplitude increases as the shear modulus increases

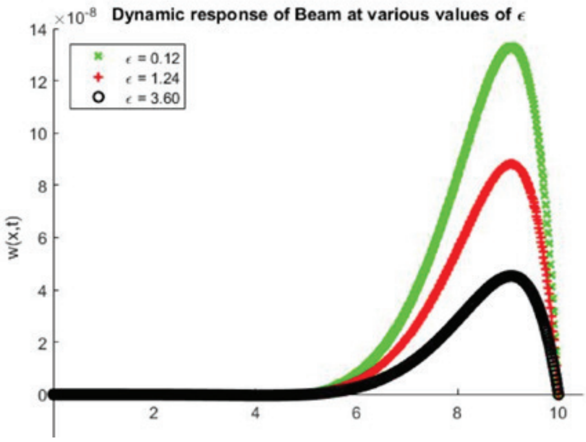

Figure 6: The effect of ϵ on the deflection of beam

Figure 6 shows the dynamic response of the beam at various values of the length of the load. It is observed that the vibration of the beam initially moves in a steady state before deflecting and the deflected amplitude decreases as the length of the load increases.

CONCLUSION

Dynamic response of a Timoshenko beam was considered in this paper work. The governing equation of fourth-order partial differential equation was reduced to a fourth-order ordinary differential equation by normalizing the governing equation. The reduced fourth-order equation was then reduced to a first order ordinary differential system and was solved using fourth-order Runge-Kunta Method. The deflection for various parameters of the beam was considered and was plotted against x using a computer program (MATLAB).

It can be concluded from Figure 4.1 - 4.6 that the vibration of the beam initially moves in a steady state before deflecting and the deflected amplitude increases as the Shear modulus, distance covered by the load, Mass of the load, Shear coefficient increases but decreases as the length of the load, coefficient of the Shear foundation increases.

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