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COMPUTATION OF ENERGY AND TENSOR PRODUCT OF GRAPHS

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ABSTRACT
Graph energy has increased tremendously, and many versions of the energy have been conceived and proposed. The products of connected graphs play a vital role in the study of DNA analysis. In this paper, the adjacency energy of the tensor product, strong product, Cartesian product, and categorial product of a complete graph K_n and a cycle graph C_n, were computed. The products $K_3 \otimes C_4, K_3 \otimes C_3, K_4 \otimes C_4, K_3 \boxtimes C_4, K_4 \boxtimes C_4, K_3 \times C_4, K_3 \times C_3, and\ K_4 \times C_4$ were considered. The adjacency matrix and associated eigenvalues were computed, and the relationship between the graph energies was established.

Keywords: Energy of a graph, Tensor product, Cartesian product, Categorial product, Strong product

INTRODUCTION

Spectral graph theory gives an expression of the combinatorial properties of a graph *G* using the eigenvalues and eigenvectors of matrices associated with the graph. This was first introduced by Gutman and Polansky (1986) to prove Cheeger's inequality for finding a sparse cut. Besides graph-theoretic research on the relationship between structural and spectral properties of graphs, another major area of research interest was quantum chemistry. However, the connections or relationships between these two aspects of work were not discovered until much later. In the Hükel Molecular Orbital (HMO) theory, the total π-electron energy, ε_π, is a quantum-chemical characteristic of conjugated molecules that agree with their thermodynamic properties (Cvetkovic *et al*., 2009). For conjugated hydrocarbons in their ground electronic states, the $\varepsilon\pi$ is calculated from the eigenvalues of the associated adjacency matrix of the molecular graph given by,

$$\varepsilon\pi = n\alpha + \varepsilon\beta \qquad (1)$$

where *n* is the number of carbon atoms, α and β are the HMO carbon-atom coulomb and carbon-atom resonance integrals (Gutman and Polansky, 1986; Cvetkovic, *et al* 2009). For the majority conjugated π-electrons system, the eigenvalues of adjacency matrix is obtain as:

$$\varepsilon = \sum_{i=1}^{n} |\lambda_i| \qquad (2)$$

where $\lambda_i (i = 1, 2, \cdots n)$ are the eigenvalues of the adjacency matrix, of the underlying molecular graph (Gutman, 1978). This special invariant defined by Gutman (1978) is called the energy of a graph denoted by ε. And in molecular structure, researchers are usually interested on ε, as it is traditionally considered as the total π-electron energy expressed in β-units.
Spectral graph theory uses the concept of linear algebra and matrix theory to investigate the properties of graphs Cvetkovic, *et al*, (2009). In the past few years, interest in graph energy has increased tremendously, and many versions of these energies have been conceived and proposed. Zhou and Bu (2012), defined Laplacian energy of a graph as the sum of the absolute derivatives of the eigenvalues of its Laplacian matrix. Furthermore, Zhou *et al*. (2018), proposed a general matrix and Laplacian energy of a simple graph with *n* vertices and *m* edges. The (n, M, m)-type bounds for the Laplacian energy was considered for the Cartesian product of two graphs (Pouyandeh *et al*., 2019). George *et al*. (2023) introduced the tensor product, the restricted tensor product, the strong product, and the restricted strong product of soft graphs. They went further to prove that these products of soft graphs are also soft graphs, and derived formulas for finding vertex count, edge count, and the sum of part degrees in them. Arshad *et al*. (2024) on the other hand, worked on Cartesian product of cycle graphs with path graphs and the energy implications. Ramane and Maraddi (2018) expressed the eigenvalues of the degree subtraction adjacency matrix of subdivision graph, semi-total point graph, semi-total line graph and total graph of a regular graph in terms of the adjacency eigenvalues of a graph G, and obtained the degree subtraction adjacency energy of these graphs. Kumar *et al*. (2024), worked on the well-known eigenvalue-based molecular descriptors such as the energy of graph E(G) and Estrada index. They test their predictive potential of some physicochemical properties of polycyclic aromatic hydrocarbons using a data set of benzenoid hydrocarbons, and further stabilized a relation of these two indices with some well-known degree-based topological indices such as the first and second Zagreb index. In this article, we shall compute the energies of tensor and strong product of a complete graph and cycle graph, and establish some inequalities.

MATERIALS AND METHODS

a. Adjacency matrix: The adjacency matrix $A(G)$ of a graph G with vertices $v_1, v_2, v_3, \cdots, v_n$ is an $n \times n$ matrix, $[a_{ij}]$ such that, $a_{ij} = 1$ if v_i is adjacent to v_j, and 0 otherwise.

b. Eigenvalue of a graph: The eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of a graph G, is eigenvalues of it's adjacency matrix.

c. Energy of a graph: The energy, E(G), of a simple graph G is defined to be the sum of the absolute values of the eigen values of G, (Gutman, (1978)).

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \quad (3)$$

d. Tensor product of two graphs: The tensor product $G \times H$ is a graph such that the vertex set of $G \times H$ is the Cartesian product $V(G) \times V(H)$; and vertices (g, h) and (g', h') are adjacent in $G \times H$ if and only if g is adjacent to g' in G and h is adjacent to h' in H.

e. Cartesian product of two graphs: The Cartesian product of two graphs $\boldsymbol{G = (VG, EG)}$ and $\boldsymbol{H = (VH, EH)}$ is a graph denoted by $\boldsymbol{G \square H}$. The vertex set of $\boldsymbol{G \square H}$ is the Cartesian product $\boldsymbol{(VG \times VH)}$, meaning each vertex of $\boldsymbol{G \square H}$ is a pair $\boldsymbol{(u, v)}$ where $\boldsymbol{u \in VG}$ and $\boldsymbol{v \in VH}$.

f. Strong product of two graphs: The strong product of $\boldsymbol{G \boxtimes H}$ of graphs G and H is a graph such that the vertex set of $\boldsymbol{G \boxtimes H}$ is the Cartesian product $\boldsymbol{V(G) \times V(H)}$; and the distinct vertices $\boldsymbol{(u', u')}$ and $\boldsymbol{(v', v')}$ are adjacent in $\boldsymbol{G \boxtimes H}$ if and only if: $\boldsymbol{u = v}$ and $\boldsymbol{u'}$ is adjacent to $\boldsymbol{v'}$, or $\boldsymbol{u}$ is adjacent to v and $\boldsymbol{u'}$ is adjacent to $\boldsymbol{v}$. It is the union of the Cartesian product and the tensor product of graphs (George *et al*, 2023).

g. Categorical product of two graphs: The Categorical product of two graphs $\boldsymbol{G = (VG, EG)}$ and $\boldsymbol{H = (VH, EH)}$ is denoted as $\boldsymbol{G \times H}$. The vertices of $\boldsymbol{G \times H}$ are the Cartesian product $\boldsymbol{(VG \times VH)}$.

RESULTS AND DISCUSSION

Energy of Tensor Products

The Adjacency matrix of the tensor product in Figure 1 is computed to be

$$A(K_2 \otimes C_4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with the characteristic polynomial $\xi(K2 \otimes C4) = x^8 - 8x^6 + 16x^4$. The eigenvalues is computed as follows:

$$|A - \lambda I_8| = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 0$$

We have $\lambda_1 = 2, \lambda_2 = 2, \lambda_3 = -2, \lambda_4 = -2, \lambda_5 = 0, \lambda_6 = 0, \lambda_7 = 0, \lambda_8 = 0$. From Equation 2.1, we have the energy of $K_2 \otimes C_4$ to be

$$E(K_2 \otimes C_4) = \sum_{i=1}^{8} |\lambda_i| = 8 \quad (4)$$

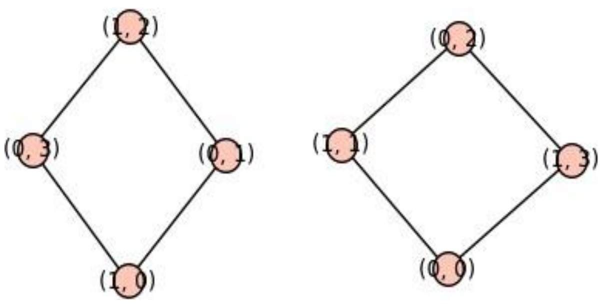

Figure 1: Tensor product of $K2$ and $C4$

The tensor product of $K3$ and $C3$ is constructed to be:

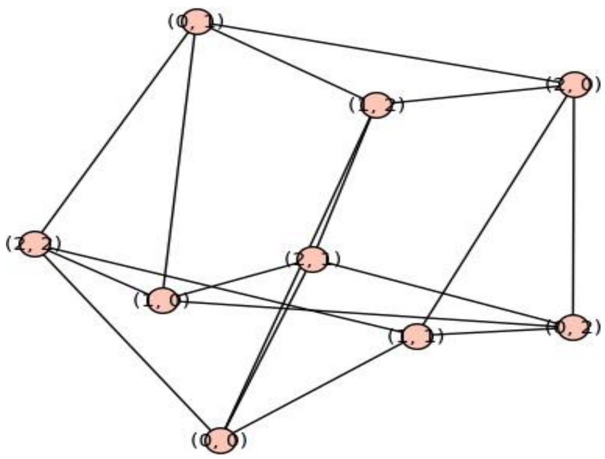

Figure 2: Tensor product of $K3$ and $C3$

The characteristic polynomial of Figure 2 is computed to be,

$$\xi(K3 \otimes C3) = x^9 - 18x^7 - 12x^6 + 81x^5 + 36x^4 - 168x^3 + 144x - 64$$

We have the eigenvalues: $\lambda_1 = 4, \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 1, \lambda_6 = \lambda_7 = \lambda_7 = \lambda_8 = \lambda_9 = -2$ Therefore, So, from Equation 2, we have $E(G) = 16$.

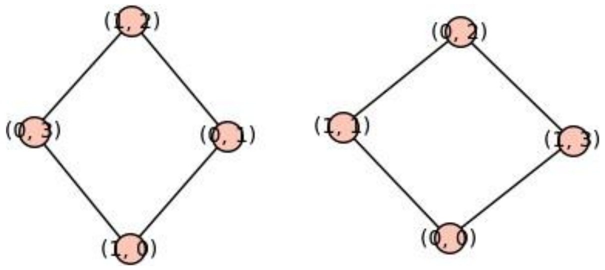

Figure 3: Tensor product of K_4 and C_4

The characteristic polynomial of Figure 3 is computed to be:

$$\xi(K4 \otimes C4) = x^{16} - 48x^{14} + 480x^{12} - 179x^{10} + 230x^8$$

with the eigenvalues given as: $\lambda_1 = 6, \lambda_2 = -6, \ \lambda_3 = \lambda_4 = \lambda_5 = 2, \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = -2, \lambda_{10}, \dots, \lambda_{16} = 0$. Thus, from equation (2.1), the $E(K_4 \otimes C_4) = 24$.

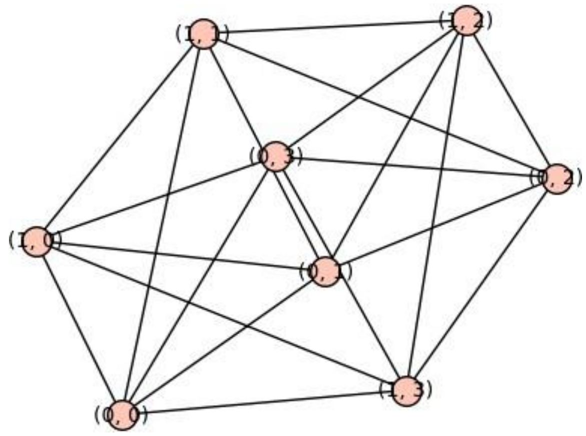

Figure 4: Strong product of K_2 and C_4

Energy of Strong Products

The Adjacency matrix of the tensor product of Figure 3.4 is computed to be

$$A(K_2 \boxtimes C_4) = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \end{pmatrix} \quad (3.2)$$

with the characteristic polynomial given as:

$$\xi(K2 \boxtimes C4) = x^8 - 20x^6 - 32x^5 + 22x^4 + 64x^3 + 12x^2 - 32x - 15$$

The eigenvalues are obtained and given as: $\lambda_1 = 5, \lambda_2 = -3, \ \lambda_3 = \lambda_4 = 1, \ \lambda_5 = \lambda_7 = \lambda_8 = -1$

Thus, $E(K_2 \boxtimes C_4) = 14$.

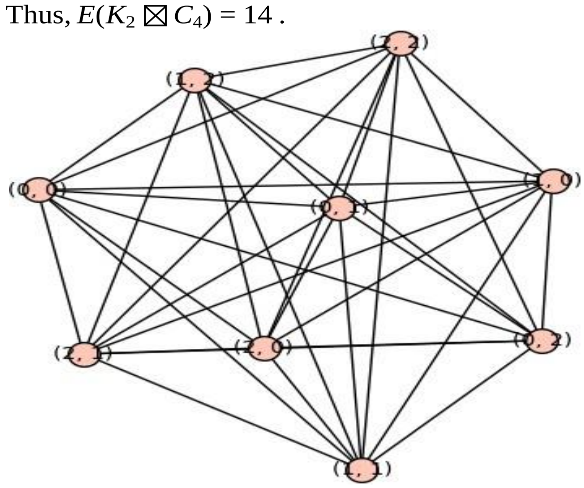

Figure 5: Strong product of K_3 and C_3

The characteristic polynomial is computed to be:

$$\xi(K3 \boxtimes C3) = x^9 - 18x^7 - 12x^6 + 81x^5 + 36x^4 - 168x^3 + 144x - 64$$

The eigenvalues is computed to be; $\lambda_1 = 8, \lambda_2 = , \dots, = \lambda_9 = -1$.

Thus, $E(K_3 \boxtimes C_3) = 16$.

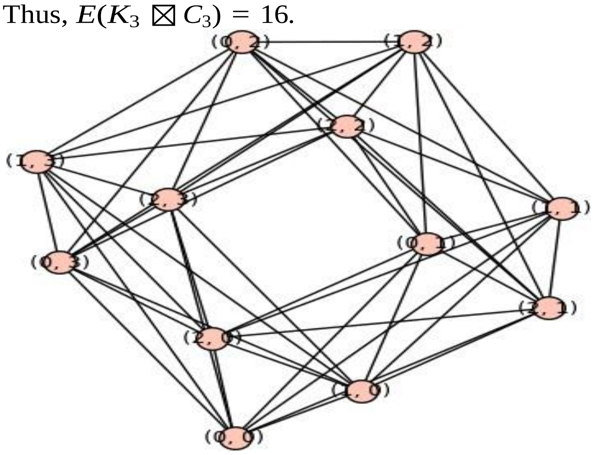

Figure 6: Strong product of K_4 and C_4

The eigenvalues of Figure 6 is computed to be $\lambda_1 = 11, \lambda_2 = \lambda_3 = 3, \lambda_4, \dots, = \lambda_{15} = -1, \lambda_{16} = -5$. Thus, the energy of $K_4 \boxtimes C_4$ is obtained to be 34.

Energy of Categorical Products

The Adjacency matrix of the Categorical product of Figure 3.8 is computed to be

$$A(K_2 \times C_4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with the characteristic polynomial $\xi(K2 \times C4) = x^8 - 8x^6 + 16x^4$

Thus, from Equation 3, $E(K_2 \times C_4) = 8$.

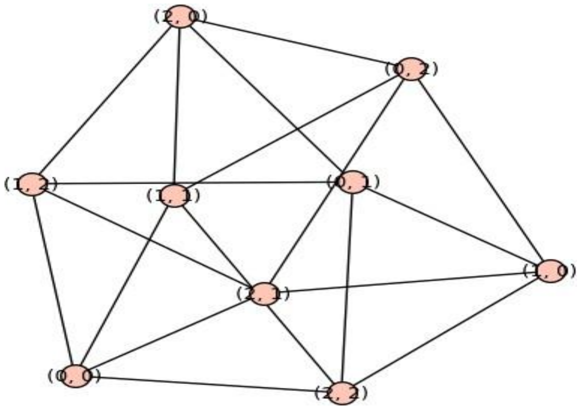

Figure 7: Categorical product of K_3 and C_3

The characteristic polynomial of $K_3 \times C_3$ is computed to be

$$\xi(K3 \otimes C3) = x^9 - 18x^7 - 12x^6 + 81x^5 + 36x^4 - 168x^3 + 144^x - 64$$

with the eigenvalues $\lambda_1 = 4, \lambda_2, \dots, = \lambda_5 = 1, \lambda_6, \dots, = \lambda_9 = -2$. From Equation 1, we have $E(G) = 16$.

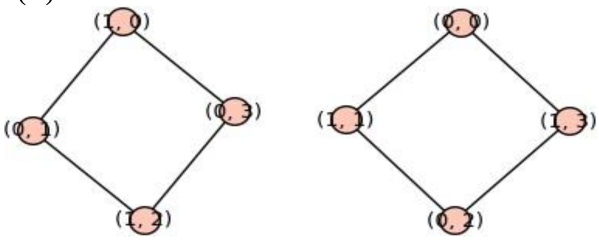

Figure 8: Categorial product of K_2 and C_4

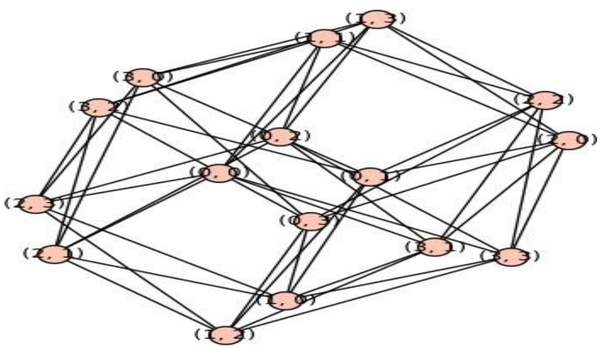

Figure 9: Categorical product of K_4 and C_4

The eigenvalues of Figure 8 computed to be: $\lambda_1 = 6, \lambda_2, \dots, = \lambda_4 = 2, \lambda_5, \dots, = \lambda_{12} = 0, \lambda_{16} = -2$. Thus, $E(K4 \times C4) = 24$.

Energy of Cartesian Product

The Adjacency matrix of the Cartesian product of $K_2 \cap C_4$ is computed to be

$$A(K_2 \square C_4) = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with the characteristic polynomial $\xi(K_2 \otimes C_4) = x^8 - 8x^6 + 16x^4$

The eigenvalues is computed as follows: $\lambda_1 = 3, \lambda_2, .., = \lambda_4 = 1, \lambda_5, ..., = \lambda_7 = 1, \lambda_8 = -3$
Thus, $E(K_4 \square C_4) = 12$.

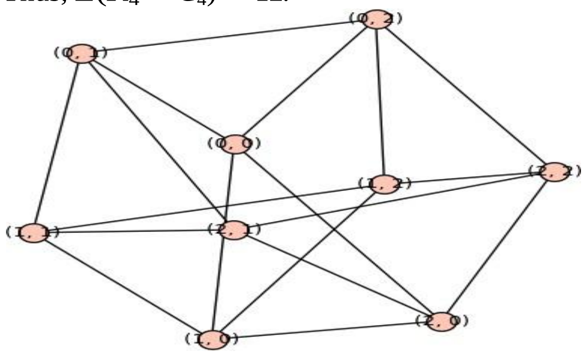

Figure 10: Cartesian product of K_3 and C_3.

The characteristic polynomial of Figure 10 is computed to be

$$\xi(K3 \square C3) = x^9 - 18x^7 - 12x^6 + 81x^5 + 36x^4 - 168x^3 + 144x - 64$$

We have the eigenvalues: $\lambda_1 = 4, \lambda_2, .., = \lambda_5 = 1, \lambda_6, ..., = \lambda_9 = -2$. From Equation 3, we have $E(G) = 16$.

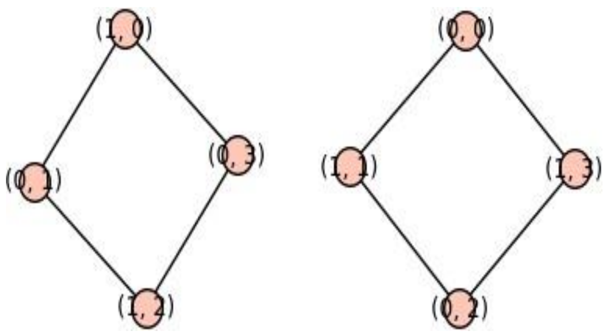

Figure 11: Cartesian product of K_2 and C_4

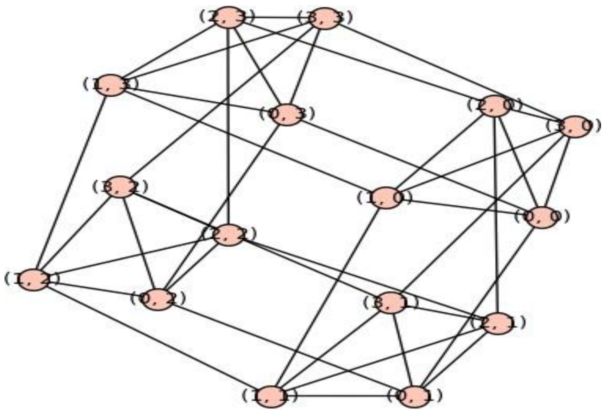

Figure 12: Cartesian product of K_4 and C_4.

The eigenvalues for the Cartesian product of Figure 12 is computed to be

$$\lambda_1 = 5, \lambda_{2,3} = 3, \lambda_5, .., = \lambda_8 = 1, \lambda_9, ..., = \lambda_{14} = -1, \lambda_{15,16} = -3$$

Thus, $E(K4 \square C4) = 27$.

Discussion

Tensor products computation:

The energy of the tensor products of $K_2 \otimes C_4$ (Figure 1), $K_3 \otimes C_3$ (Figure 2) and $K_4 \otimes C_4$ (Figure 3) was computed. It was observed that the energy increases as the number of vertices of the complete graph increases. Their characteristic polynomials are polynomial of degree 8, 16, and 24, respectively which are all equal to the number of vertices in the various graphs.

$$E(K3 \otimes C3) = 2E(K2 \otimes C4)$$

The energy of Km⊗Cn is roughly proportional to m because the eigenvalues of Km dominate. Tensor products "combine" the structures multiplicatively in the spectrum, which explains why:
E(K3⊗C3)=2E(K2⊗C4) even though the graphs themselves have different structures.

The General formula (approximate) is give as: E(Km⊗Cn)=∑i=1m∑j=1n|λi(Km)·μj(Cn)|

Cartesian products computation:

The spectra radius of $K4 \square C4$ (Figure 9) is greater than that of $K3 \square C3$ (Figure 5), which is also greater than that of $K2 \square C4$ (Figure 4), that is,

$$\rho(K4 \square C4) = \rho(K3 \square C3) + 1 = \rho(K2 \square C4) + 2.$$

Strong products computation

It was also observed that the adjacency energy of the strong products of K_2 and C_4 (Figure 4) , and K_4 and C_4 (Figure 6) is greater than that of their tensor products. We have that

$E(K2 \boxtimes C4) = E(K2 \otimes C4) + 6.$ And $E(K4 \boxtimes C4) = E(K4 \otimes C4) + 10 = E(K4 \otimes C4) + E(K2 \otimes C4) + 2.$

For strong product**,** the adjacency matrix is sum of Cartesian + tensor matrices in general, their spectra do not simply add**,** but energy roughly increases because more edges are added**.**

Categorical products computation:
The energy of the Categorical products and the tensor product of K_2 and C_4 (Figure 11 and 1) are equal. Thus,

$$E(K2 \times C4) = E(K2 \otimes C4) = 8.$$

CONCLUSION

The adjacency energy of the tensor product, strong product, cartesian product and categorial product of a complete graph K_n and a cycle graph C_n was computed. The products: $K3 \otimes C4, K3 \otimes C3, K4 \otimes C4, K3 \boxtimes C4, K3 \boxtimes C4, K4 \boxtimes C4, K3 \times C4, K3 \times C3 \text{ and } K4 \times C4$ are considered. The adjacency matrix and associated eigenvalues were computed and the relationship between the graph energies was established. Therefore, the energy of these graph products of are equal K_n and C_n for $n = 3$, that is, when the number of vertices of both graph products is 3. Consequently, t he result of this research can be used to study the DNA analysis.

Conflict of Interest: The author declares no conflict of interest.

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