PHYSICAL SCIENCES

ON THE COMBINATORIAL PROPERTIES OF NILPOTENT AND IDEMPOTENT CONJUGACY CLASSES OF THE INJECTIVE ORDER-DECREASING TRANSFORMATION SEMIGROUP

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ABSTRACT

The elements of the injective order-decreasing transformation semigroup using path structure (Circuits and Proper paths) introduced by Stephen Lipscomb is presented. The elements were grouped according to their conjugacy class, the nilpotents and the idempotents were identified and the nilpotent and idempotent conjugacy class chain decompositions studied. Some observations were made and their order enumerated. Formulae were found for their order and combinatorial properties

Keywords: *Combinatorial properties, Nilpotent conjugacy classes, Idempotent conjugacy classes, Injective Order-decreasing Transformation semigroup*

INTRODUCTION

Semigroup of injective order-decreasing transformations, a subsemigroup of the symmetric inverse semigroup (I_n) was introduced by (Umar, 1992) and defined as thus:

Let α be a transformation InI_n, is said to be of order-decreasing if $(\forall x \in dom\, \alpha) x\alpha \leq x$. The order of the injective order-decreasing transformations (ID_n) corresponds to sequence A000110: 1, 2, 5, 15, 52, 203, 877, 4140 … of the online Encyclopedia of integer sequences (OEIS). A formula for its order is given as $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$ where B_k is the k-th Bell's number. It is well known that an element $x \in I_n$ is nilpotent $(x^n = 0)$ for some $n > 0$. A property of nilpotent element among others is $x\alpha \neq x \,\forall\, x \in dom\alpha$.

In this paper, we are adopted the path notations invented by (Lipscomb,1996) for I_n which he defined as follows: Let $\mathbb{N} = x_1, x_2, \ldots, x_m$ and $\alpha \in I_n$ have domain $d\alpha = x_1, x_2, \ldots, x_m$ and if $x_1\alpha = x_2, x_2\alpha = x_3, x_{m-1}\alpha = x_m, x_m = y$.

Then α is a path. Having a circuit or a proper path depends on the value of y. If $y = x_1$ then $\alpha = (x_1, x_2, \ldots, x_m)$ is a circuit of length m. If $y \neq x_1$ then $\alpha = (x_1, x_2, \ldots, x_m, y]$ is a proper path of length (m + 1)

Various text or papers have slightly different path notations as can be seen in (Munn, 1957) where he used the notations "links" and "cycles" for proper paths and circuits respectively. He would write $(12)(345] \in I_5$ as $(12)[345]$ where (12) is the cycle while [345] is the link. (Gomes and Howie, 1987) denoted a primitive nilpotent as "||1, 2, …, m||" while (Sullivan, 1987) in his study of semigroups generated by nilpotent transformations denoted a proper path of length (m + 1) as m-chains [1, 2, …, m + 1] and a circuit of length m as m-cycles (1, 2, …, m)

Let G be a group. An element $x \in G$ is said to be conjugate to an element $y \in G$ if there exists $g \in G$ such that $y = g^{-1}xg$. Since conjugation is important to group theory, it was only quite natural to have extended it to some certain classes of semigroups.

Theorem 1

Let $x, y \in I_n$. Then the following holds:

a. x is conjugate to y if and only if they have the same path structure
b. x is nilpotent if and only if its path structure are joins of only proper paths.
c. x is idempotent if and only if all the paths in its path structure is of length one

Proof:

a. Let $a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & - \end{pmatrix} \in I_6 = (123)(456]$

and

$$b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & - & 5 & 6 & 4 \end{pmatrix} \in I_6 = (123](456).$$

a and b has a circuit of length 3 and a proper 3-path and hence falls in the same conjugacy class. Also we can find a permutation g *in permutation group* such that $y = g^{-1}xg$ or $gy = xg$ by matching the paths in a vertical order

$$a = (123)(456]$$
$$b = (456)(123]$$

from where we can deduce $g = (14)(25)(36)$

b. Let φ be a proper path and σ a circuit. A circuit is an extension of a permutation and permutation group has no nilpotent, therefore $x = \varphi\sigma \Rightarrow x^n \neq 0$. Hence x is nilpotent *iff* its path structure are only proper paths.

c. Assuming φ *or* σ has length 2, $x\alpha \neq x \,\forall x \in dom\alpha$.

This implies that $x\alpha^2 \neq x\alpha$.

The definition of Conjugacy in arbitrary semigroup seems not to be unique as can be seen in(Kudryavtseva and Mazorchuk,2009) where they compared three approaches of Conjugacy on semigroups. (Dauns, 1989) gave a definition for monoid and (Lallement, 1979) for free semigroups.

MONOGENIC SUBSEMIGROUP

Let S be a semigroup and let $a \in S$, then the monogenic subsemigroup $\langle a \rangle$ consists of all elements of S that can be expressed as positive integral powers of a. Here we say that a is a generating set of S. In this paper, we only consider the finite monogenic subsemigroup. If repetitions occur in the positive powers of a, then we have that $a^m = a^m + a^r = a^t$ where m is called the index, r the period and $m < t$. We have that the powers $a, a^2, \ldots, a^m, \ldots, a^{t-1}$ are distinct and therefore *order of* $a = m + r - 1$. Below is a monogenic subsemigroup generated by

$$a = (12](34567)(8) \in I_8$$

$$a = (12](34567)(8);\ a^2 = (1](2](35746)(8);$$
$$a^3 = (1](2](36475)(8);\ a^4 = (1](2](37654)(8);$$
$$a^5 = (1](2](3)(4)(5)(6)(7)(8);\ a^6 = (1](2](34567)(8);$$
$$a^7 = (1](2](35746)(8)$$

From the illustration above, we can see that $a^2 = a^2 + a^5 = a^7$. It is worthy to note that the term 'Monogenic' was introduced by (Howie, 1995) as against the term 'cyclic' used by (Clifford and Preston, 1961) and (Lipscomb,1996).Howie claimed that it did not merit the name 'cyclic' because it is not always cyclic as in the case of a singleton generator.

Theorem2

Let $x \in I_n$. Then the following holds

a. The index of x is the maximum of lengths of all the proper paths in it. The index is one if no proper path exist in x.

b. The period of x is the lowest common multiple of all the lengths of the circuits in x.

Proof: See (Lipscomb, 1996) pp 13.

MATERIALS AND METHODS

The conjugacy classes of $\alpha \in ID_n$ were arranged according to the fix of α denoted as $f(\alpha)$ and defined by $f(\alpha) = |F(\alpha)| = |\{x \in X_n : x\alpha = x\}|$ for any number of ID_n. The nilpotent conjugacy classes are marked as (*), while the idempotent conjugacy classes are marked as (@). The Index and period of each conjugacy class was also found. Some illustrations are given below

Table 1: Conjugacy classes of ID_1

$f(\alpha)$	Conjugacy classes	Index	Period
0	(1]*@	1	1
1	(1)@	1	1

Table 2: Conjugacyclasses of ID_2

$f(\alpha)$	Conjugacy classes	Index	Period
0	(1](2]*@	1	1
	(21]*	2	1
1	(1)(2]@	1	1
2	(1)(2)@	1	1

Table 3: Conjugacy classes of ID_3

$f(\alpha)$	Conjugacy classes	Index	Period
0	(1](2](3]*@	1	1
	(21](3]*	2	1
	(321]*	3	1
1	(1)(2](3]@	1	1
	(1)(32]	2	1
2	(1)(2)(3]@	1	1
3	(1)(2)(3)@	1	1

Table 4: Conjugacy classes of ID_4

$f(\alpha)$	Conjugacy classes	Index	Period
0	(1](2](3](4]*@	1	1
	(21](3](4]*	2	1
	(21](43]*	2	1
	(321](4]*	3	1
	(4321]*	4	1
1	(1)(2](3](4]@	1	1
	(1)(32](4]	2	1
	(1)(432]	3	1
2	(1)(2)(3](4]@	1	1
	(1)(2)(43]	2	1
3	(1)(2)(3)(4]@	1	1
4	(1)(2)(3)(4)@	1	1

RESULTS AND DISCUSSION

We observed some combinatorial relations between numbers associated with the nilpotent and idempotent conjugacy classes of ID_n. We define the following numbers:

$N_n =$ the cardinality of nilpotent conjugacy classes of ID_n.

$E_n =$ the cardinality of idempotent conjugacy classes of ID_n.

$M_n =$ the total number of chains in the nilpotent conjugacy class chain decomposition of ID_n.

$L_n =$ the total number of chains in the idempotent conjugacy class chain decomposition of ID_n.

Table 5: Table of combinatorial relations

n	N_n	E_n	M_n	L_n
1	1	2	1	2
2	2	3	3	4
3	3	4	5	6
4	5	5	7	8
5	7	6	9	10
6	11	7	11	12
7	15	8	13	14
8	22	9	15	16
9	30	10	17	18
10	42	11	19	20

The following results are products of Table 5 above.

Lemma 1: Let $\alpha \in ID_n$, the nilpotent conjugacy classes of ID_n is given as $|E_n| = n + 1$.

Proof: For each idempotent rank of $n = 0, 1, \ldots, n$ in ID_n, there exist at least one idempotent element. Also idempotent elements of a particular rank fall under a conjugacy class. Thus we have $n + 1$ dempotent conjugacy classes

Lemma 3: The total number of chains in the nilpotent conjugacy class chain decomposition of ID_n is $M_n = 2n - 1$.

Proof: Let α be a nilpotent transformation in ID_n with domain $X_n = 1, 2, \ldots, n$.

The number of images $(x\alpha) < X_n$ in any *n* of ID_n Therefore number of total nilpotent conjugacy class chain decomposition of *n* will be $(n) empty\ maps + (n-1)\ maps$ of other combinations of chain decomposition of n. Hence $M_n = 2n - 1$

Lemma 4: The total number of chains in the idempotent conjugacy class chain decomposition of ID_n is $L_n = 2n$

Proof: From Theorem 1.1 we have that x is idempotent if and only if all the paths in its path structure is of length one. It implies that it is either a circuit of length one or a proper 1-path of n. Generally, there are two kinds of path in n ways. Therefore $L_n = 2n$

CONCLUSION

The sequence of $N_n = 1, 2, 3, 5, 7, 11, 15, 22, 30, 45, \ldots$ $where\ n = 1, 2, \ldots$ is the partitions of a positive integer A000041 of the Online Encyclopedia of Integer Sequences (OEIS). Also the idempotent and nilpotent conjugacy classes for ID_n as well as the total number of chains in the idempotent conjugacy class chain decomposition of ID_n has been shown and hence satisfies the part structure of length one.

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