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CONSTRUCTION OF SUITABLE LYAPUNOV FUNCTIONS FOR SYSTEMS OF LINEAR AND NONLINEAR DIFFERENTIAL EQUATIONS WITH STABILITY OF THEIR TRIVIAL SOLUTIONS.

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ABSTRACT
In this paper, suitable Lyapunov functions are constructed for systems of linear and nonlinear differential equations. The Lyapunov functions constructed are tested for the stability of the trivial solutions for the stated problems and the results show that the trivial solution of the linear system of differential equations is asymptotically stable while the trivial solution of the nonlinear system of differential equations is stable (in the sense of Lyapunov). The scientific implication of these results is that the solutions starting near the zero solution for asymptotic stability eventually result to the zero solution in a limit as while solutions starting near the zero solution for Lyapunov stability stay close to the zero solution.

Keywords: *Lyapunov functions, Stability, trivial solutions, linear and non linear differential equations.*

INTRODUCTION

In the theory of ordinary differential equations, Lyapunov functions are scalar functions that are used to prove the stability of equilibrium points or zero stability of ordinary differential equations. For many classes of ordinary differential equations, the existence of Lyapunov functions is a necessary and sufficient condition for stability.

In a research work by Ademola and Arawomo, 2011, they examined the asymptotic behaviour of solutions of third order nonlinear differential equations. Also, Afuwape and Omeike, 2010 examined the stability and boundedness of solutions for third order delay differential equations. In another work, Andreev, 1997, considered the problem of asymptotic stability of non autonomous functional differential equations under the supposition that the derivative of Lyapunov functional is a non negative scalar function. The results obtained modified and extended a number of well known results. The sum of square decomposition method was used by Parachristodou and Prajna, 2002 to construct Lyapunov function for system with equality, inequality and integral constraints which allows certain non polynomials and non linearity in the vector field to be analysed.

In another paper, Gil, 1977 formulated explicit stability for non-linear retarded system with separate autonomous linear parts in terms of the roots of the characteristic polynomials based on the recent estimates for the matrix resolvent. He discussed the global stability; considered the estimates for the norm of the Green's function and then derived a bound for a region of attraction for the zero solution.

In this paper, suitable Lyapunov functions are constructed for systems of linear and non-linear differential equations. The Lyapunov functions constructed will then be tested for the stability of trivial solution for the stated problems.

MATERIALS AND METHODS

This research work considers the second order, linear, ordinary differential equation of the form

$$\ddot{x} + a\dot{x} + bx = 0 \qquad (1)$$

where $a, b, > 0, x \neq 0$ and the nonlinear second order ordinary differential equation of the form

$$\ddot{x} + \dot{x}f(x) + h(x) = 0 \qquad (2)$$

where $f(x) > 0$, $h(x)$ >0 are functions of x with $f(0) = h(0) = 0$

Definition

Let $\Omega \in \Re^n$ be a neighbourhood of the origin and let $V : \Omega \to \Re$. Then, the function $V = V(x)$ is said to be:

(i)positive definite if $V(0) = 0$ and $V(x) > 0$ for $0 \neq x \in \Omega$

(ii) negative definite if $V(0) = 0$ and $V(x) < 0$ for $0 \neq x \in \Omega$

(iii)positive semidefinitie if $V(0) = 0$ and $V(x) \geq 0$ for $0 \neq x \in \Omega$

(iv)negative semidefinite if $V(0) = 0$ and $V(x) \leq 0$ for $0 \neq x \in \Omega$

Theorem

Consider the continuous autonomous system

$$\dot{x} = f(x) , \ f(0) = 0 \qquad (3)$$

where $f : \Re^n \to \Re^n$ is such that solution exists and unique for a given initial data. The trivial solution $x \equiv 0$ of (3) is said to be:

(i)stable(in the sense of Lyapunov) if $\exists$ a function $V : \grave{U} \to \Re$, $\Omega \subset \Re^n$ that is negative definite and whose derivative $\dot{V}(x) = \frac{dV(x(t))}{dt} = \sum_{i=1}^{n} \frac{\partial V(x(t)) f_i(x(t))}{\partial x_i}$ with respect to the system (3) is positive semi definite or $V : \Omega \to \Re$, $\Omega \subset \Re^n$ is positive definite and whose derivative $\dot{V}(x)$ with respect to the system (3) is negative semi definite

(ii) asymptotically stable if $\exists$ a function $V : \Omega \to \Re$, $\Omega \subset \Re^n$ that is negative definite and whose derivative

$\dot{V}(x) = \frac{dV(x(t))}{dt} = \sum_{i=1}^{n} \frac{\partial V(x(t)) f_i(x(t))}{\partial x_i}$ $\dot{V}(x)$ is positive definite or $V : \Omega \to \Re, \Omega \subset \Re^n$ is positive definite and whose derivative $\dot{V}(x)$ with respect to the system (3) is negative definite.

Exact solutions

Starting with the linear second order ordinary differential equation (1), its equivalent system of first order equations is

$$\dot{x} = y, \quad \dot{y} = -ay - bx \quad ....4$$

We seek a function $V : \Re^2 \to \Re$ such that

i) $V(x, y)$ is positive definite

ii) $\dot{V}(x, y)$ is positive or negative semi definite.

Now consider for (1),

$$2V(x, y) = k_1x^2 + k_2y^2 + 2k_3xy \quad ....(5)$$

Differentiating (5) gives

$$\dot{V}(x, y) = (k_1 - ak_3 - bk_2)xy + (k_3 - ak_2)y^2 - bk_3x^2$$

There are three cases for the stability of trivial solution of (1)

Case 1: $\dot{V} \alpha x^2$

Here, $k_1 - ak_3 - bk_2 = 0$

$k_3 - ak_2 = 0$ and $bk_3 > 0$

Hence, $k_3 = ak_2, k_1 = a^2k_2 + bk_2$ and $abk_2 > 0$ .Then,

fix $k_2 \equiv 1, k_3 = a, k_1 = a^2 + b$ where $a, b > 0$

Then, (5) becomes

$$2V_1(x, y) = (a^2 + b)x^2 + y^2 + 2axy = bx^2 + (ax + y)^2 > 0 \quad ......(7)$$

which is positive definite since $a, b > 0$ and $x, y \neq 0$ .Then, differentiating (7)

$$\dot{V}_1(x, y) = -abx^2 < 0 \quad ....(8)$$

which is negative semi definite since $a, b > 0$ and $\dot{V}_1(0, \varepsilon) = 0$

Case 2: $\dot{V} \alpha y^2$

. Here $k_1 - ak_3 - bk_2 = 0,$ $bk_3 = 0$ and $k_3 - ak_2 < 0$

Assume $b \neq 0$, then $k_3 = 0$ so that $k_1 = bk_2$.

Then, fix $k_2 \equiv 1$, $k_1 = b$ and using (4)

$$2V_2(x,y) = bx^2 + y^2 > 0 \quad \text{.....................(9)}$$

which is positive definite since and . Then differentiating (9) to obtain

$$\dot{V}_2(x,y) = -ay^2 < 0 \quad \text{.........................(10)}$$

which is negative semi definite since $a > 0$ and

$\dot{V}_2(\varepsilon, 0) = 0$ which is negative semi definite since and

Case 3: $\dot{V} \alpha \left(x^2 + y^2\right)$

To conclude the stability of the trivial solution of

(1), $2V_3(x,y) = 2V_1(x,y) + 2V_2(x,y)$

$$2V_3(x,y) = 2bx^2 + (ax+y)^2 + y^2 > 0 \quad \text{............(11)}$$

which is positive definite since $a, b > 0$ and $x, y \neq 0$ .Hence, (11) is the suitable Lyapunov function for the system of linear ordinary differential equation (1). Differentiating (11) gives,

$$\dot{V}_3(x,y) = -a\left(bx^2 + y^2\right) < 0 \quad \text{......12}$$

which is negative definite since and . This shows that the trivial solution of the system of (1) is asymptotically stable.

This result is then extended to the system of non-linear second order ordinary differential equation (2), the equivalent system of first order equations is

$$\dot{x} = y, \ \dot{y} = -yf(x) - h(x) \quad \text{..............(13)}$$

From equations (1) and (2), it is observed that

$a = f(x)$ and $bx = h(x)$ .Then, using the three cases of the linear system above, the three cases for the non linear system are given below:

Case 1: $\dot{V} \alpha x^2$

$$2V_1^*(x,y) = 2\int_0^x h(s)ds + [xf(x) + y]^2 > 0 \quad \text{...... (7*)}$$

which is positive definite since $x, y \neq 0$ and $xf(x), xh(x) > 0$

Differentiating (7*) gives

$$2\dot{V}_1^*(x,y) = 2\dot{x}h(x) + 2(xf(x) + y)(\dot{x}f(x) + xf'(x) - yf(x) - h(x))$$

$$\dot{V}_1^*(x,y) = -xf(x)[(x+y)f'(x) + h(x)] \quad \text{....................(8*)}$$

which is negative semi definite since $\dot{V}_1^*(0,\varepsilon) = 0$

Case 2: $\dot{V} \alpha y^2$

$$2V_2^*(x,y) = 2\int_0^x h(s)ds + y^2 > 0 \quad \text{............../(9*)}$$

which is positive definite since $x, y \neq 0$ and $xh(x) > 0$

Differentiating (9*) gives

$$2\dot{V}_2^*(x,y) = 2\dot{x}h(x) + 2y\dot{y}$$

$$\dot{V}_2^*(x,y) = -y^2 f(x) \quad \text{................(10*)}$$

which is negative semi definite since $\dot{V}_2^*(\varepsilon, 0) = 0$

Case 3 $\dot{V} \alpha \left(x^2 + y^2\right)$

To conclude the stability of the trivial solution for (2),

$$2V_3^*(x,y) = 2V_1^*(x,y) + 2V_2^*(x,y)$$

$$2V_3^*(x,y) = 4\int_0^x h(s)ds + [xf(x) + y]^2 + y^2 > 0 \quad \text{.........(11*)}$$

which is positive definite since $x, y \neq 0$ and $xf(x), xh(x) > 0$

Equation (11*) is the Lyapunov function for the non linear system.

Then, differentiating (11*) gives

$$2\dot{V}_3^*(x,y) = 2\dot{V}_1^*(x,y) + 2\dot{V}_2^*(x,y)$$

Hence, $\dot{V}_3^*(x,y) = f(x)\{x(x+y)f'(x) + xh(x) + y^2\}$ ...(12*)

which is negative semi definite since $f(0) = h(0) = 0$

and $\dot{V}_3^*(0,\varepsilon) = 0$

RESULTS AND DISCUSSION

Lyapunov functions have been obtained for systems of linear and nonlinear ordinary differential equations. The Lyapunov function obtained for the linear system in case 3 (is a combination of the Lyapunov functions obtained in cases 1 and 2) is positive definite and its derivative is negative definite. Hence, the trivial solution of the linear system is asymptotically stable. On the other hand, the Lyapunov function obtained for the nonlinear system in case 3* (is the combination of the Lyapunov functions obtained in cases 1* and 2*) is positive definite and its derivative is negative semi definite. Hence, the trivial solution of the nonlinear system is stable (in the sense of Lyapunov) .

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