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AN IMPROVED FAMILY OF BLOCK METHODS BASED ON THE EXTENDED TRAPEZOIDAL RULE OF SECOND KIND AND THEIR APPLICATIONS

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ABSTRACT

We present a family of three and five step extended trapezoidal rule of second kind (ETR_2s) in block form for thesolution of stiff ordinary differential equations. The block methods were derived via interpolation and collocation procedures. We determined the order, error constant, zero stability to show convergence. The absolute stability region of our block methodsindicates that they are A-Stable, hence suitable for stiff system of ODE problems. The solution curves obtained tends to suggest that our methods compete favorably with the well-known ODE Solver ODE23s. Four numerical examples were used to demonstrate the efficiency of the new block methods,the absolute errors attest to the performance of our methods in-terms of accuracy.

Keywords: *Stiff Ordinary Differential Equations, Extended Trapezoidal Rule, General Linear Methods, Matrix Finite Difference, Stability Polynomial.*

INTRODUCTION

In science and engineering, mathematical models are usually developed to help in the understanding of physical phenomena. Those models often yield equations that contain some derivative of an unknown function(s) of one or several variables. Such equations are called differential equations. Differential equations do not only arise in the physical sciences but also in diverse fields as economics, medicine, psychology, operations research and even in areas such as biology and anthropology. These equations have received much attention in the last 20 years (Shampine and Watts,1969).

Interestingly, some differential equations arising from the modeling of physical phenomena often do not have theoretical solutions. Hence, the development of numerical methods to obtain approximate solutions becomes necessary.

The problem of obtaining starting values for linear multistep methods with step numbers when solving differential equations numerically has been the challenge faced by applied mathematicians(especially numerical analyst). In the past, one step methods were used in obtaining starting values for such methods. The method proposed in this paper as in (Henrici,1962) does not share the disadvantage of not been self-starting.

The study considers the derivation of a family of block extended trapezoidal rule of second kind (BETR$_2$s) for odd step numbers $k=3$ and 5 for the numerical solution of (1) from the same continuous formulation to overcome the problem of starting values. Our proposed methods are derived based on the interpolation and collocation procedures(Atkinson, 1989; Gladwell and Sayers, 1976; Lie and Norsett,1989; Onumanyi *et al.*,1994). Block methods were first introduced by Milne, 1953for used only as a means of obtaining starting values for predictor-corrector algorithms and has since then been developed by several researchers (Rosser, 1967; Sarafyan,1965; Shampine and Watts, 1969)for general purpose.

The solution of equation (1) has extensively been studied by various researchers (Atkinson, 1989; Butcher,1965, 2003;Brugnano, 1998; Chu and Hamilton, 1987; Dahlquist,1963; Fatunla, 1991; Gladwelland Sayers, 1976; Graggand Stetter, 1964;Henrici, 1962;Lambert, 1973, 1991; Lieand Norset, 1989; Milne, 1953; Onumanyi *et al.*, 1994; Rosser, 1967;Sarafyan, 1965;Shampine and Watts, 1969;Yao and Akinfenwa, 2011).

We considerthe numerical solution of the stiff initial value problem (ivp) of the first order ordinary differential equations (ODEs) of the form:

$$y' = f(x,y)\ ,\ y(a) = y_0 \ .... \quad (1)$$

We seek a solution in the range $a \le x \le b$ where and are finite and we assume that f satisfies that the conditions which guarantees that (1) has a unique continuously differentiable solution which we shall indicate by $Y(x)$ .Consider the sequence of points $\{x_n\}$ define by $x_n = a + \boldsymbol{h}$ $, n = 0,1,\dots,b-n/h$ where the parameter is a constant step size.

The continuous formulation of a k-step linear multistep method for the solution of (1) of our method is in the form

$$\bar{y} = \sum_{j=0}^{k-1} \alpha_j(x) y(x_{n+j}) + h\left[\beta_v(x) f_{n+v} + \beta_{v-1}(x) f_{n+v-1}\right] \ ....(2)$$

where $k = 2v-1$, is the step number and $\alpha_j(x)$ $j = 0,1,\dots,k-1,$ $\beta_v(x)$ $\beta_{v-1}(x)$ are the continuous coefficients of the method to be determined. Equation (2) generates a main discrete method when evaluated at endpoints and with some additional methods when evaluated at interior points, these methods are combine and implemented as a block, which simultaneously yields approximations to our desired problem(s).

MATERIALS AND METHODS

Derivation of the three step block ETR$_2$s (BETR$_2$s 1) Method

Case I: $k=3$

Then the continuous formulation (2) of BETR$_2$s 1becomes

$$\bar{y} = \alpha_0(x) y_n + \alpha_1(x) y_{n+1} + \alpha_2(x) y_{n+2} + h\left[\beta_1(x) f_{n+1} + \beta_2(x) f_{n+2}\right] \ ....(3)$$

where, $\alpha_0(\xi) = \left(\frac{-3\xi}{h} + \frac{13}{4}\frac{\xi^2}{h^2} - \frac{3}{2}\frac{\xi^3}{h^3} + 1 + \frac{1}{4}\frac{\xi^4}{h^4} \right)$

$$\alpha_1(\xi) = \left(\frac{4\xi^2}{h^2} - \frac{4\xi^3}{h^3} + \frac{\xi^4}{h^4} \right)$$

$$\alpha_2(\xi) = \left(\frac{3\xi}{h} - \frac{29}{4}\frac{\xi^2}{h^2} + \frac{11}{2}\frac{\xi^3}{h^3} - \frac{5}{4}\frac{\xi^4}{h^4} \right) \ ....(4)$$

$$\beta_1(\xi) = \left(-4\xi + \frac{8\xi^2}{h} - \frac{5\xi^3}{h^2} + \frac{\xi^4}{h^3} \right)$$

$$\beta_2(\xi) = \left(-\xi + \frac{5}{2}\frac{\xi^2}{h} - \frac{2\xi^3}{h^2} + \frac{1}{2}\frac{\xi^4}{h^3} \right)$$

Evaluating (3), we obtained a three step block ETR_2s which can be represented in block matrix finite difference form as,

$$AY_m = BY_{m-1} + h\{CF_m + DF_{m-1}\} \quad \text{.................(5)}$$

where

$$A = \begin{pmatrix} -\frac{9}{12} & \frac{9}{12} & \frac{1}{12} \\ \frac{24}{2} & -\frac{27}{2} & 0 \\ 0 & \frac{3}{2} & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & \frac{1}{12} \\ 0 & 0 & -\frac{3}{2} \\ 0 & 0 & \frac{3}{2} \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{17}{2} & -\frac{14}{2} & \frac{1}{2} \\ \frac{4}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

and

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

and

$$Y_m = (y_{n+1}, y_{n+2}, y_{n+3})^T,$$

$$Y_{m-1} = (y_{n-2}, y_{n-1}, y_n)^T$$

$$F_m = (f_{n+1}, f_{n+2}, f_{n+3})^T$$

$$F_{m-1} = (f_{n-2}, f_{n-1}, f_n)^T \quad \text{.................(6)}$$

Derivation of the five step block ETR_2s ($BETR_2s$ 2) Method

Case II: $k = 5$

Then the continuous formulation (2) of $BETR_2s$ 2 can be express as:

$$\bar{y} = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_2(x)y_{n+2}$$
$$+ \alpha_3(x)y_{n+3} + \alpha_4(x)y_{n+4} +$$
$$h[\beta_2(x)f_{n+2} + \beta_3(x)f_{n+3}] \quad \text{.......................... (7)}$$

Where, we generate the unknown coefficients of the methods as:

$$\alpha_0(\xi) = \left(-\frac{35\xi}{12h} + \frac{121}{36}\frac{\xi^2}{h^2} - \frac{95}{48}\frac{\xi^3}{h^3} + \frac{91}{144}\frac{\xi^4}{h^4} - \frac{5}{48}\frac{\xi^5}{h^5} + 1 + \frac{1}{144}\frac{\xi^6}{h^6} \right)$$

$$\alpha_1(\xi) = \left(-\frac{1}{12}\frac{\xi^6}{h^6} + \frac{12\xi}{h} - \frac{23\xi^2}{h^2} + \frac{52}{3}\frac{\xi^3}{h^3} - \frac{77}{12}\frac{\xi^4}{h^4} + \frac{7}{6}\frac{\xi^5}{h^5} \right)$$

$$\alpha_2(\xi) = \left(-\frac{1}{4}\frac{\xi^6}{h^6} + \frac{9\xi}{h} - \frac{105\xi^2}{4h^2} + \frac{28\xi^3}{h^3} - \frac{27}{2}\frac{\xi^4}{h^4} + \frac{3\xi^5}{h^5} \right)$$

$$\alpha_3(\xi) = \left(\frac{11}{36}\frac{\xi^6}{h^6} - \frac{52}{3}\frac{\xi}{h} + \frac{395}{9}\frac{\xi^2}{h^2} - \frac{124}{3}\frac{\xi^3}{h^3} + \frac{659}{36}\frac{\xi^4}{h^4} - \frac{23}{6}\frac{\xi^5}{h^5} \right)$$

$$\alpha_4(\xi) = \left(\frac{1}{48}\frac{\xi^6}{h^6} - \frac{3}{4}\frac{\xi}{h} + \frac{2\xi^2}{h^2} - \frac{97}{48}\frac{\xi^3}{h^3} + \frac{47}{48}\frac{\xi^4}{h^4} - \frac{11}{48}\frac{\xi^5}{h^5} \right)$$

$$\beta_2(\xi) = \left(-\frac{1}{4}\frac{\xi^6}{h^5} + 18\xi - \frac{87}{2}\frac{\xi^2}{h} + \frac{155}{4}\frac{\xi^3}{h^2} - \frac{65}{4}\frac{\xi^4}{h^3} + \frac{13}{4}\frac{\xi^5}{h^4} \right)$$

$$\beta_3(\xi) = \left(-\frac{1}{6}\frac{\xi^6}{h^5} + 8\xi - \frac{62}{3}\frac{\xi^2}{h} + \frac{20\xi^3}{h^2} - \frac{55}{6}\frac{\xi^4}{h^3} + \frac{2\xi^5}{h^4} \right)$$

....(8)

Evaluating (7), we obtained a five step block ETR_2s in the form:

$$H\Psi_m = J\Psi_{m-1} + h\{L\Phi_m + N\Phi_{m-1}\} \quad \text{...............(9)}$$

where

$$H = \begin{pmatrix} -\frac{15}{120} & -\frac{80}{120} & \frac{80}{120} & \frac{15}{120} & -\frac{1}{120} \\ -\frac{512}{24} & -\frac{2592}{24} & \frac{2692}{24} & \frac{381}{24} & 0 \\ -\frac{16}{24} & -\frac{108}{24} & \frac{80}{24} & \frac{43}{24} & 0 \\ -\frac{28}{24} & 0 & \frac{28}{24} & \frac{1}{24} & 0 \\ -\frac{144}{24} & -\frac{108}{24} & \frac{208}{24} & \frac{9}{24} & 0 \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{120} \\ 0 & 0 & 0 & 0 & -\frac{35}{24} \\ 0 & 0 & 0 & 0 & -\frac{1}{24} \\ 0 & 0 & 0 & 0 & \frac{1}{24} \\ 0 & 0 & 0 & 0 & -\frac{35}{24} \end{pmatrix}$$

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}$$ and

$$L = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{167}{2} & \frac{157}{2} & 0 & \frac{1}{2} \\ 0 & \frac{6}{2} & \frac{8}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{18}{2} & \frac{8}{2} & 0 & 0 \end{pmatrix}$$ and

$$\Psi_m = \left(y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, y_{n+5}\right)^T$$
$$\Psi_{m-1} = \left(y_{n-4}, y_{n-3}, y_{n-2}, y_{n-1}, y_n\right)^T$$
$$\Phi_m = \left(f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}, f_{n+5}\right)^T$$
$$\Phi_{m-1} = \left(f_{n-4}, f_{n-3}, f_{n-2}, f_{n-1}, f_n\right)^T$$

ORDER, CONSISTENCY, ZERO-STABILITY, ABSOLUTE STABILITY REGION OF THE BLOCK ETR$_2$S

Definition 3.1: Consistency
The block ETR$_2$s (5) and (7) are said to be consistent if the order of the individual method is greater or equal to one, that is, if

(i) $\rho(1) = 0$, and

(ii) $\rho'(1) = \sigma(1)$

where $\rho(z)$, and $\sigma(z)$ are respectively the 1st and 2nd characteristic polynomials(Lambert,1973).

Definition 3.2: Zero-Stability
The block ETR$_2$s (5) and (7) are said to be zero-stable if the roots

$$\rho(\lambda) = \det\left[\sum_{i=0}^{k} A^{(i)}\lambda^{k-i}\right] = 0 \quad ....(10)$$

Satisfies $|\lambda_j| \le 1, j = 1,...,k$ and for those roots with $|\lambda_j| = 1,$ the multiplicity does not exceed two (Lambert, 1973; 1991).

Definition3.3: (Dahlquist, 1963)
A numerical method is said to be A-stable if its region of absolute stability contains the whole of the left-hand half complex plane $\mathrm{Re}\, h\lambda < 0$

Definition3.4:A (α) -Stable
A numerical method is said to be A(α)-Stable, $\alpha \in (0, \frac{\pi}{2})$, if its region of absolute stability contains the infinite wedge $w_\alpha = [h\lambda - \alpha < \pi - \arg h\lambda]$

Definition 3.5: Convergence of LMM

The necessary and sufficient conditions of all LMM are that it must be consistent and zero-stable (Dahlquist, 1963).

Order of the block ETR$_2$s
According toHenrici (1962),the LMM is said to be of order p if $c_0 = c_1 = c_2 = ...c_p = 0,$, we extend this approach to determine the order of the entire block method which can be expressed as:

$$\sum_{i=0}^{k} \alpha_{ij} y_{n+j} = h\sum_{i=0}^{k} \beta_{ij} f_{n+j} \quad ....(11)$$

where $j = 0,1,...,k$, is a positive integer, equation (11) can be expanded to give the following system of equation

$$\begin{pmatrix} \alpha_{01} & \alpha_{11} & \alpha_{21} & . & . & . & \alpha_{k1} \\ \alpha_{02} & \alpha_{12} & \alpha_{22} & . & . & . & \alpha_{k2} \\ \alpha_{03} & \alpha_{13} & \alpha_{23} & & & & \alpha_{k3} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \alpha_{0k} & \alpha_{1k} & \alpha_{2k} & . & . & . & \alpha_{kk} \end{pmatrix} \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ . \\ . \\ y_{n+k} \end{pmatrix} = h \begin{pmatrix} \beta_{01} & \beta_{11} & \beta_{21} & . & . & . & \beta_{k1} \\ \beta_{02} & \beta_{12} & \beta_{22} & . & . & . & \beta_{k2} \\ \beta_{03} & \beta_{13} & \beta_{23} & & & & \beta_{k3} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \beta_{0k} & \beta_{1k} & \beta_{2k} & . & . & . & \beta_{kk} \end{pmatrix} \begin{pmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ . \\ . \\ f_{n+k} \end{pmatrix}$$

......................(12)

The expression (12) is equivalent to (11)

where $\vec{\alpha}_0 = \begin{pmatrix} 0 \\ 2 \\ 3 \\ . \\ . \\ . \\ 0k \end{pmatrix}, \quad \vec{\alpha}_1 = \begin{pmatrix} 11 \\ 12 \\ 13 \\ . \\ . \\ . \\ 1k \end{pmatrix}, ..., \quad \vec{\alpha}_k = \begin{pmatrix} k1 \\ k2 \\ k3 \\ . \\ . \\ . \\ kk \end{pmatrix},$

and $\vec{\beta}_0 = \begin{pmatrix} 0 \\ 2 \\ 3 \\ . \\ . \\ . \\ 0k \end{pmatrix}, \quad \vec{\beta}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ . \\ . \\ . \\ 1k \end{pmatrix}$ üü $\vec{\beta}_k = \begin{pmatrix} k1 \\ k2 \\ k3 \\ . \\ . \\ . \\ k \end{pmatrix}$

Adopting the order procedure used in the single case for the block method, we recall that

$$L_h y(x) = \sum_{i=0}^{k} [\bar{\alpha}_j y(x+jh) - h\bar{\beta}_j f(x+jh, y(x+jh))]$$

..............................(13)

where $y(x)$, is the exact solution satisfying (1). Carrying out Taylor series expansion on (13) about x yields the equation

$$L_h y(x) = \vec{c}_0 y(x) + \vec{c}_1 h y'(x) +$$

$$\vec{c}_2 h^2 y''(x) + \ldots + \vec{c}_q h^q y^{(q)}(x)$$ ....(14)

$$\vec{c}_0 = \begin{pmatrix} c_{01} \\ c_{02} \\ c_{03} \\ \cdot \\ \cdot \\ \cdot \\ c_{0p} \end{pmatrix}, \quad \vec{c}_1 = \begin{pmatrix} c_{11} \\ c_{12} \\ c_{13} \\ \cdot \\ \cdot \\ \cdot \\ c_{1p} \end{pmatrix} \text{üü} \quad \vec{c}_p = \begin{pmatrix} c_{p1} \\ c_{p2} \\ c_{p3} \\ \cdot \\ \cdot \\ \cdot \\ c_{p} \end{pmatrix}$$ ....(15)

The block linear multistep method is said to be of order p if $\vec{c}_0 = \vec{c}_1 = \vec{c}_2 = \ldots \vec{c}_p = 0, \vec{c}_{p+1} \neq 0$
And the local truncation error is expressed as

$$T_n = \vec{c}_{p+1} h^{p+1} y^{(p+1)}(x_n)$$ ....(16)

Absolute Stability Region of block LMM
To determined the absolute stability region of the block method, they are reformulated into General Linear Methods of Burage and Butcher (1980) where they used as partition $(s+r)(s+r)$ matrix containing A, B, U and V expressed in the form

$$\begin{bmatrix} Y \\ y^{i-1} \end{bmatrix} = \begin{bmatrix} A & U \\ B & V \end{bmatrix} \begin{bmatrix} hf(Y) \\ y^{i} \end{bmatrix}, i = 1, 2, \ldots, N$$ ....(17)

Where the matrices A, B, U and V are substituted into a stability matrix

$$M(z) = V + zB(I - zA)^{-1}U$$ ....(18)

Which is in-turn substituted into a stability function

$$\rho(\lambda, z) = \det(\lambda I - Mz)$$ ....(19)

In a MATLAB code to obtained the stability region of the method.

RESULTS AND DISCUSSION

Order of the Three Step Block ETR_2s Method
Using equations (11), (12),…,(16), our block ETR_2s (5) yields

$$\vec{\alpha}_0 = \begin{pmatrix} -\frac{1}{12} & \frac{3}{2} & -\frac{3}{2} & 0 \end{pmatrix}^T$$

$$\vec{\beta}_0 = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}^T$$

$$\vec{\alpha}_1 = \begin{pmatrix} -\frac{9}{12} & \frac{24}{2} & 0 & 0 \end{pmatrix}^T$$

and

$$\vec{\beta}_1 = \begin{pmatrix} \frac{1}{2} & -\frac{17}{2} & \frac{4}{2} & 0 \end{pmatrix}^T$$

$$\vec{\alpha}_2 = \begin{pmatrix} \frac{9}{12} & -\frac{27}{2} & \frac{3}{2} & 0 \end{pmatrix}^T$$

$$\vec{\beta}_2 = \begin{pmatrix} \frac{1}{2} & -\frac{14}{2} & \frac{1}{2} & 0 \end{pmatrix}^T$$

$$\vec{\alpha}_3 = \begin{pmatrix} \frac{1}{12} & 0 & 0 & 0 \end{pmatrix}^T$$

$$\vec{\beta}_3 = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}^T$$

then

$$\vec{c}_0 = \sum_{j=0}^{k} \vec{\alpha}_j = \sum_{j=0}^{3} \vec{\alpha}_j = \vec{\alpha}_0 + \vec{\alpha}_1 + \vec{\alpha}_2 + \vec{\alpha}_3$$

$$= \begin{pmatrix} -\frac{1}{12} \\ \frac{3}{2} \\ -\frac{3}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{9}{12} \\ \frac{24}{2} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{9}{12} \\ -\frac{27}{2} \\ \frac{3}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{12} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{c}_1 = \sum_{j=0}^{k} j\vec{\alpha}_j - \vec{\beta}_j = \sum_{j=0}^{3} j\vec{\alpha}_j - \vec{\beta}_j = (\vec{\alpha}_1 + 2\vec{\alpha}_2 + 3\vec{\alpha}_3) - (\vec{\beta}_0 + \vec{\beta}_1 + \vec{\beta}_2 + \vec{\beta}_3)$$

$$= \left\{ \begin{pmatrix} -\frac{9}{12} \\ \frac{24}{2} \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} \frac{9}{12} \\ -\frac{27}{2} \\ \frac{3}{2} \\ 0 \end{pmatrix} + 3\begin{pmatrix} \frac{1}{12} \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} - \left\{ \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ -\frac{17}{2} \\ \frac{4}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ -\frac{14}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} \right\} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{c}_2 = \sum_{j=0}^{3} \frac{1}{2!}\left(j^2\vec{\alpha}_j\right) - j\vec{\beta}_j = \frac{1}{2!}\left(\vec{\alpha}_1 + 2^2\vec{\alpha}_2 + 3^2\vec{\alpha}_3\right) - \left(\vec{\beta}_1 + 2\vec{\beta}_2 + 3\vec{\beta}_3\right)$$

$$= \frac{1}{2!}\left\{\begin{pmatrix} -\frac{9}{12} \\ \frac{24}{2} \\ 0 \\ 0 \end{pmatrix} + 2^2\begin{pmatrix} \frac{9}{12} \\ -\frac{27}{2} \\ \frac{3}{2} \\ 0 \end{pmatrix} + 3^2\begin{pmatrix} \frac{1}{12} \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\} - \left\{\begin{pmatrix} \frac{1}{2} \\ -\frac{17}{2} \\ \frac{4}{2} \\ 0 \end{pmatrix} + 2\begin{pmatrix} \frac{1}{2} \\ -\frac{14}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}\right\} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{c}_5 = \sum_{j=0}^{3} \frac{1}{5!}\left(j^5\vec{\alpha}_j\right) - \frac{1}{4!}\left(j^4\vec{\beta}_j\right) = \frac{1}{5!}\left(\vec{\alpha}_1 + 2^5\vec{\alpha}_2 + 3^5\vec{\alpha}_3\right) - \frac{1}{4!}\left(\vec{\beta}_1 + 2^4\vec{\beta}_2 + 3^4\vec{\beta}_3\right)$$

$$= \frac{1}{5!}\left[\begin{pmatrix} -\frac{9}{12} \\ \frac{24}{2} \\ 0 \\ 0 \end{pmatrix} + 2^5\begin{pmatrix} \frac{9}{12} \\ -\frac{27}{2} \\ \frac{3}{2} \\ 0 \end{pmatrix} + 3^5\begin{pmatrix} \frac{1}{12} \\ 0 \\ 0 \\ 0 \end{pmatrix}\right] - \frac{1}{4!}\left[\begin{pmatrix} \frac{1}{2} \\ -\frac{17}{2} \\ \frac{4}{2} \\ 0 \end{pmatrix} + 2^4\begin{pmatrix} \frac{1}{2} \\ -\frac{14}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} + 3^4\begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}\right] = \begin{pmatrix} \frac{1}{5!} \\ -\frac{1}{3!} \\ -\frac{1}{5.2!3!} \\ 0 \end{pmatrix}$$

We have shown that the order of the block method (5) is $p = 4$ with error constant $T_5 = (\frac{1}{5!} \ -\frac{1}{3!} \ -\frac{1}{5.2!3!})^T$ and by definition (3.1), the method is consistent.

Zero-Stability of the Three Step Block Method

The block method (5) can be expressed in the form

$$A^{(0)} y_{m+1} = \sum_{i=1}^{k} A^{(i)} y_{m-1} + h\sum_{i=0}^{k} \beta^{(i)} f_{m+1} \quad \text{................(20)}$$

Where h is a fixed mesh size $A^{(i)}, \beta^{(i)}, i = 0,1,2,...,k$ are $r \times s$ identity matrix and y_m, y_{m+1} and y_{m-1} are vectors of numerical estimates. When (5) is expressed as (20), the values $A^{(0)}$, $A^{(1)}$, $\beta^{(0)}$ and $\beta^{(1)}$ are obtained as

$$A^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A^{(1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\beta^{(0)} = \begin{pmatrix} \frac{9}{24} & \frac{-5}{24} & \frac{1}{24} \\ \frac{4}{3} & \frac{1}{3} & 0 \\ \frac{9}{8} & \frac{9}{8} & \frac{3}{8} \end{pmatrix} \quad \text{and} \quad \beta^{(1)} = \begin{pmatrix} 0 & 0 & \frac{3}{8} \\ 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{3}{8} \end{pmatrix}$$

Substituting $A^{(0)}$ and $A^{(1)}$ into (10) gives the first characteristic polynomial of (5) as

$$\rho(\lambda) = \det(\lambda A^{(0)} - A^{(1)}) = 0$$

$$= \det\left[\lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \det\begin{bmatrix} \lambda & 0 & -1 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda - 1 \end{bmatrix}$$

$$= \lambda^2(\lambda - 1) \Rightarrow \lambda_1 = 1, \lambda_2 = \lambda_3 = 0$$

By definition (3.2), the block method (5) is zero-stable. Hence by definition (3.5), our block method is convergent.

Absolute Stability Region of the Three Step block ETR$_2$s Method

We transform (5) into (17), where A, B, U and V are given as

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{17}{24} & -\frac{14}{24} & \frac{1}{24} \\ \frac{1}{3} & \frac{4}{3} & \frac{1}{3} & 0 \\ 0 & 12 & 12 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} -9 & 9 & 1 \\ 0 & 0 & 1 \\ \frac{17}{24} & 0 & -\frac{3}{24} \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 12 & 12 & 0 \\ \frac{1}{3} & \frac{4}{3} & \frac{1}{3} & 0 \\ 0 & -\frac{17}{24} & -\frac{14}{24} & \frac{1}{24} \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & 0 & 1 \\ \frac{17}{24} & 0 & -\frac{3}{24} \\ 0 & 0 & 1 \\ -9 & 9 & 1 \end{pmatrix} \quad \text{.......................... (21)}$$

We plotted the region of absolute stability of the General Linear Method (21) by substituting the values of A, B, U and V into (18) and then into the stability function (19). Then using a MATLAB code, we obtained the desired region

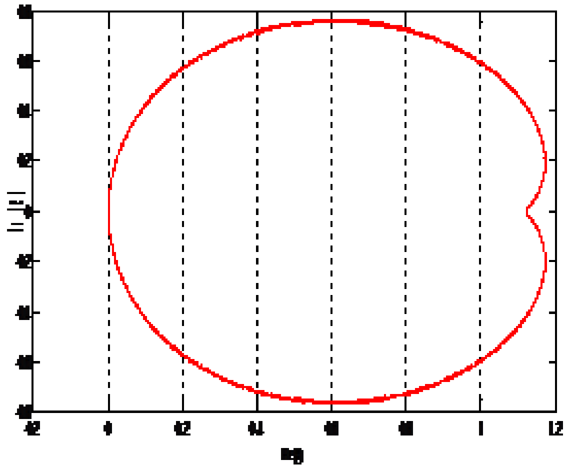

Fig. 1: Stability region of the Three Step block ETR_2s (5)

Order of the Five Step Block ETR_2s Method

Similarly, applying (11),(12),…,(16) on (9), yields the order of the block method (9) as $p = 6$ with error constant $T_7 = (-\frac{1}{7.5!} \; -\frac{187}{7.5!} \; -\frac{1}{3\;.3!} \; \frac{1}{7.5!} \; -\frac{1}{3\;.2!})^T$ and by definition (3.1), the method is consistent.

Zero-Stability of the Five Step Block ETR_2s Method

Applying (10) on (9), yields

$$\rho(\lambda) = \det(\lambda A^{(0)} - A^{(1)}) = 0$$

$$= \det\left[\lambda\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \det\begin{bmatrix} \lambda & 0 & 0 & 0 & -1 \\ 0 & \lambda & 0 & 0 & -1 \\ 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & \lambda - 1 \end{bmatrix}$$

$$= \lambda^4(\lambda - 1)$$

$$\Rightarrow \lambda_1 = 1, \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$$

. By definition (3.2), the block method (9) is zero-stable. Hence by definition (3.5), our block method is convergent.

Absolute Stability region of the Five Step Block ETR_2s

Similarly, equation (9) can be transformed into (17) where A, B, U and V which yields:

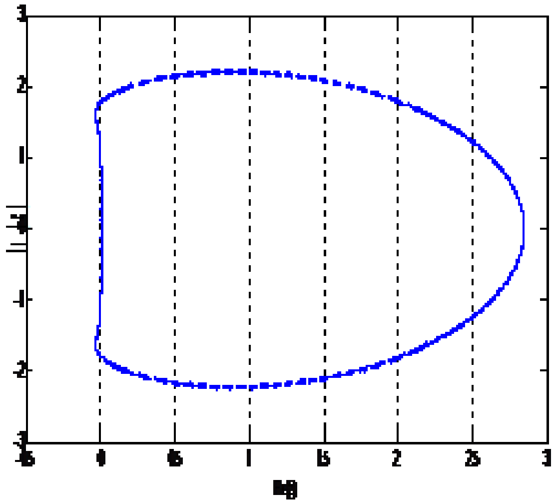

Fig. 2: Stability region of the Five Step block ETR_2s (9) Method

NUMERICAL ILLUSTRATIONS

We present four numerical results to illustrate the performance of the newly derived Block ETR_2s ($BETR_2s$) methods for step numbers k=3 and 5 for stiff system of ordinary differential equations.We also reported the performance of our new block methodsthrough their absolute errors and with the well-known MatLabode solver ode23s where the theoretical solution does not exist.

Problem 5.1: Lorenz Equations

$$y_1' = 10(y_2 - y_1)$$

$$y_2' = -y_1 y_3 + 28y_1 - y_2$$

$$y_3' = y_1 y_2 - \frac{8}{3} y_3$$

$$y_2(0) = -16.0335$$

$$y_3(0) = 24.4450$$

$$0 \le x \le 40, \qquad h = 0.02$$

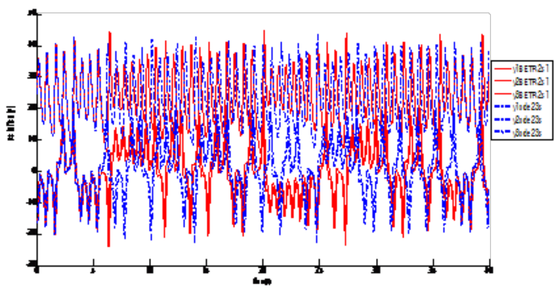

Fig. 3: Solution curve for problem 5.1 using the new Block ETR_2s method (5)

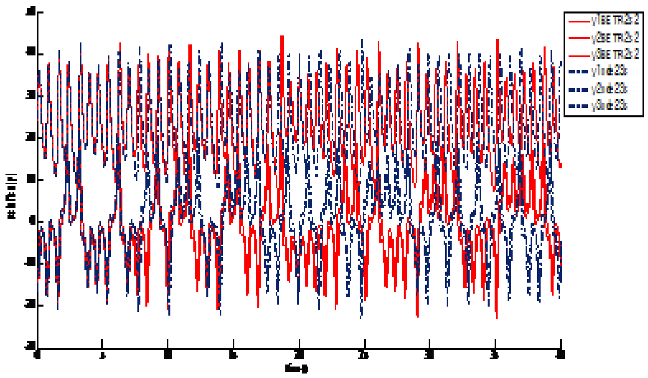

Fig. 4: Solution curve for problem 5.1 using the new Block ETR_2s method (9)

Problem 5.2: Stiff System of Ordinary Differential Equations

Consider the linear stiff system of ordinary differential equations on the range $0 \le x \le 10$.

$y_1' = -10y_1 + 50y_2$

$y_2' = -50y_1 - 10y_2$

$y_3' = -40y_3 - 200y_4$

$y_4' = -200y_3 - 40y_4$

$y_5' = -0.2y_5 - 2y_6$

$y_6' = -2y_5 - 0.2y_6 y_1(0)$

$= 0,\ y_2(0)$

$= 1,\ y_3(0)$

$= 0,\ y_4(0)$

$= 1,\ y_5(0)$

$= 0,\ y_6(0) = 1, h$

$= 0.01$

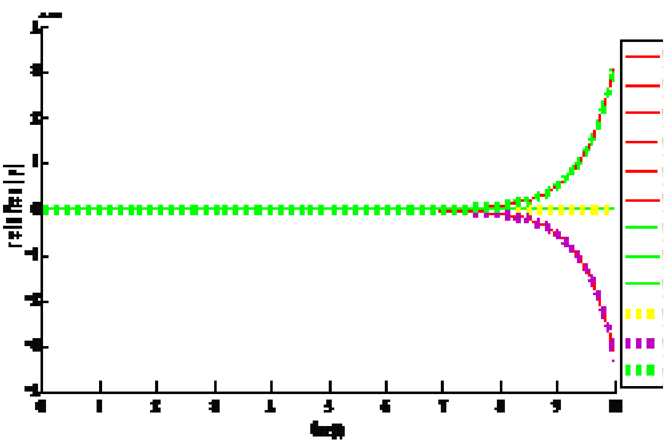

Fig. 5: Solution curve for problem 5.2 using the new Block ETR2s method (5)

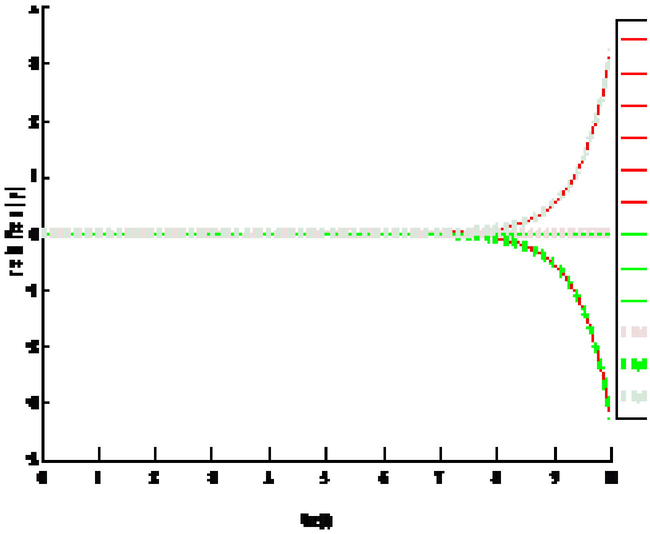

Fig. 6: Solution curve for problem 5.2 using the new Block ETR_2s method (9)

Problem 5.3: Stiff Linear System

$y_1' = -10y_1 + \beta y_2$

$y_1(0) = 1$

$y_2' = -\beta y_1 - 10y_2$

$y_2(0) = 1$

$y_3' = -\gamma y_3$

$y_3(0) = 1$

$0 \le t \le T, \qquad h = 0.001,$

$T = 1$

The theoretical solution of this problem is given by

$y_1(t) = e^{-\gamma}(\cos(\beta t) + \sin(\beta t))$

$y_2(t) = e^{-\gamma}(\cos(\beta t) - \sin(\beta t))$

$y_3(t) = e^{-\gamma}$

[NB: This problem has been extensively studied by Shampine (1977) and reported that the system is stiff when $\beta = 21$ and $\gamma = 10$].

Table 1: Absolute Errors of the First Component for Problem 5.3 using three and five Step Method

x	Block ETR_2s (5)	Block ETR_2s (9)
0	0.0000E+00	0.0000E+00
0.1	1.8353E-01	1.0037E-01
0.2	3.4263E-02	2.2921E-02
0.3	6.6503E-02	4.6796E-02
0.4	3.2783E-02	1.1983E-02
0.5	5.2758E-03	5.2087E-03
0.6	8.3778E-03	3.9261E-03
0.7	1.3255E-03	4.3560E-04
0.8	1.1492E-03	4.7368E-04
0.9	5.1294E-04	2.1774E-04
1.0	6.4821E-05	1.3117E-06

Problem 5.4: Stiff Linear System

$$y_1'(x) = -21y_1(x) + 19y_2(x) - 20y_3(x),\ y_1(0) = 1$$
$$y_2'(x) = 19y_1(x) - 21y_2(x) + 20y_3(x),\ y_2(0) = 0$$
$$y_3'(x) = 40y_1(x) - 40y_2(x) + 40y_3(x),$$
$$y_3(0) = -1,\ 0 \le x \le 10,\ h = 0.1$$

Theoretical solution given below

$$y_1(x) = \frac{1}{2}(e^{-2x} + e^{-40x}(\cos(40x) + \sin\ (40x)))$$
$$y_2(x) = \frac{1}{2}(e^{-2x} - e^{-40x}(\cos(40x) + \sin\ (40x)))$$
$$y_3(x) = \frac{1}{2}(2e^{-40x}(\sin(40x) + cos(40x)))$$

Table 2: Absolute Errors of the First Component for Problem 5.4 using Three and Five Step Method

x	Block ETR2s (5)	Block ETR2s (9)
0	0.0000E+00	0.0000E+00
0.1	3.2435E-02	1.3925E-02
0.2	1.1275 E-02	5.4352E-03
0.3	7.5309 E-03	2.1115E-03
0.4	2.5545 E-03	4.4039E-04
0.5	6.2457 E-04	6.6284E-05
0.6	1.2075 E-04	7.0568E-06
0.7	4.5362 E-06	1.0985E-07
0.8	3.0745 E-06	1.7744E-08
0.9	6.8491 E-06	1.2783E-07
1.0	4.0916 E-06	7.5504E-08

CONCLUSIONS

This paper has demonstrated the derivation of family of three and five step extended trapezoidal rule of second kind (ETR_2s) and implemented as self-stating methods in block form for the solution of stiff ordinary differential equations.Some numerical properties of the block methods were investigated and the methods were shown to be convergent with good absolute stability regions. We also demonstrated the accuracy of our newly derived block methods by considering four numerical examples.From the solution curves obtained, our method(s) competes favorably with the well-known Ode solver ode 23s.

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