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FIXED POINTS OF A NONHOMOGENIOUS SECOND ORDER DIFFERENCE EQUATION OF ACCRETIVE TYPE IN HILBERT SPACES

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ABSTRACT

In this study, we establish the existence of a common fixed point in a nonhomogeneous second order difference equation of accretive type in Hilbert spaces. We used a contraction and rational inequality in Hilbert spaces to obtain our result and its uniqueness.

Keywords: Accretive operator, Hilbert spaces, Parallelogram law, Norm spaces, Inner product

INTRODUCTION

Hilbert space is generally known as a complete inner product space; complete in the metric defined by the inner product. According to Rousan (2023), the theory of Hilbert spaces was introduced into functional analysis exactly in 1912 by David Hilbert (1862-1943). The currently used terminology in respect of Hilbert spaces is analogous to that of Euclidean geometry and was formed by Schmidt in 1908.

Neumann (1903-1957) later formulated an axiomatic of Hilbert spaces and developed the modern theory of operators in Hilbert spaces. His wonderful contribution to this sphere of study has provided the mathematical foundation of quantum mechanics. Hilbert spaces constitute at present the most important examples of Banach spaces, not only because they are the most natural and closest generalization in the realm of infinite dimensions of our classical Euclidean geometry, but principally because up to now, they are the most useful spaces in the application of functional analysis (Luca, 2017).

Studies on operators as well as accretive operators gained prominence in mathematics and mathematical sciences in recent times. In fact, in functional analysis, the study of operators is very important. An operator (a mapping) in mathematics is a continuous transformation or functional relation or abstract function. As regards accretive operators or accretive mappings, Barbu (1976), in his investigation looked at the term 'accretive' a set in Banach spaces. Then, Sari (2015), submitted that since an element of the subset of $X \times X$ (X being a Banach space) can be assumed as a pair of an element of a domain with its mapping value, it is important to tag accretive as an operator. So, an accretive operator is a multi-valued mapping.

Thus, when extended to Banach spaces, accretive operators are consisted in the study of the maps of the form $A : D(A) \subseteq X \longrightarrow 2^X$ which are said to be accretive if for each $a, b \in D(A)$ and $\lambda > 0$, $\| a - b \| \leq |a - b + \lambda(A(a) - A(b))|$ where, X is a Banach space, A is an accretive operator and $D(A)$ is the domain of A (Garcia-Falset & Morales, 2005) . Suppose that $X = H$ is a Hilbert space, the concept of accretive operator then coincides with that of monotone operator (Moosavi & Hossaini, 2024). F. Browder, T. Kato and a host of other mathematics' researchers in 1960s studied accretive operators in Banach spaces (Garcia-Falset & Morales, 2005). Barbu (1976), Poffald (1986) and Apreutesei (1998), as well as many others followed suit by studying the theory of nonlinear operators of accretive type with their corresponding dynamics in Banach spaces in 1970s, 1980s and 1990s respectively.

These studies were later transformed into findings on 'differential equations of accretive type in Banach spaces and the second order difference equations of accretive (monotone) type in Banach spaces (Jamshidnezhad & Saeidi, 2018). Gyegwe and Bassi (2024), investigated existence and uniqueness of the second order difference equation of accretive type in the framework of 2-Banach spaces.

Consider the following second order difference equation of accretive type spaces in spaces. It is a non-homogeneous difference equation of the form,

$$\begin{cases} u_{m+1} - (1 + \theta_m)u_m + \theta_m u_{m-1} \in c_m A u_m + f_m \,, 1 \leq m \\ \quad < N \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathbf{(1.1)} \\ \qquad\qquad u_0 = x, \quad \sup\{\| u_m \| : m \geq 0\ 0\} < \infty \end{cases}$$

Where A is a nonlinear m-accretive operator in a real Banach space X , $c_m > 0$ and $\theta_m > 0$ (Jamshidnezhad and Saeidi, 2018). Equation (1.1) has the following second order accretive type nonhomogeneous evolution equation equivalence:

$$\begin{cases} g(t)u^{''})(t) + h(t)u^{'}(t) \in Au(t) + f(t), \\ \qquad\qquad everywhere\ on\ \mathbb{R} \qquad\qquad \mathbf{(1.2)} \\ u(0) = u_0, \sup\{\| u(t) \| ; t \geq 0\} < \infty. \end{cases}$$

Appreutessei (2003), investigated the homogeneous case of equation (1.1) , whereby the existence, uniqueness and the asymptotic behaviour were established in Banachspaces. The asymptotic behaviour of solutions to equation (1.1) with A as maximal monotone operator in Hilbert spaces was studied by Khatibzadeh (2012).

Exploring the second order nonhomogeneous difference equation (1.1), Rouhani and Khatibzadeh (2011) as well as Apreutesei and Apreustesei (2012), who made their findings on the asymptotic behaviour of solutions(1.1) where A is a maximal monotone operator in Hilbert spaces with c_m, θ_m and f_m. This study focuses on equation (1.1) in the framework of Hilbert spaces.

MATERIALS AND METHODS

Definition 2.1: Norm and a Norm Space

A norm of a number k (realor complex denoted by $\| k \|$) is a non-negative length of the number. A norm space Xor $(X, \|.\|)$ is a vector space X over. a real or complex field $\mathbb{C}$ with a non-negative function $\|.\|$: $X \longrightarrow [0, \infty)$ such that :

(i) $\|.\| > 0$ for all $x \neq 0$ (positive definiteness),
(ii) $\| \alpha x \| = |\alpha| \| x \|$ for all $x \in X$ and $\alpha \in \mathbb{R}$
(iii) $\| x + y \| \leq |\alpha| \| x \| + \| y \|$ for all $x, y \in X$ (triangle inequality) (Moris, 2019).

Definition 2.2: Metric and Metric Spaces

Let M be a non-empty set and ρ be a real-valued function defined on $M \times M$ such that for any $x, y \in M$, the following conditions are satisfied:

(i) $\rho(x, y) \geq 0$ (non-negativity),
(ii) $\rho(x, y) = 0 \Leftrightarrow x = y$ (homogeneity),
(iii) $\rho(x, y) = \rho(y, x)$(symmetry)
(iv) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all $x, y \epsilon M$ (triangle inequality).

Then ρ is said to be a metric on M. $(M, \rho)$is known as the metric space and $\rho(x, y)$ is called the distance between x and y (Teschl, 2019).

Definition 2.3: Inner Product and Inner Product Space

Let X be a complex vector space. A mapping $\langle .,. \rangle$: $X \times X \longrightarrow \mathbb{C}$ is known as an inner product in X if for all $a, b \in X$ and $\alpha \in \mathbb{C}$, the following axioms are satisfied:

(i) $\langle a, b \rangle = \overline{\langle a, b \ \rangle}$ (the bar refers to complex conjugate);
(ii) $\langle \alpha a + \beta b, \ c \rangle = \alpha \langle a, c \rangle + \beta \langle b, c \rangle$;
(iii) $\langle a, b \rangle \geq 0$;
(iv) $\langle a, a \rangle = 0 \Leftrightarrow 0 \ a = 0$

An inner product space or pre-Hilbert space is a vector space X or $(X, \langle . , . \rangle)$ with inner product defined on X (Debnath & Mikusinski, 2005).

Definition 2.4: Accretive Operator

An accretive operator A with domain $D(A)$ in a complex Hilbert space H is said to be accretive if Re $\langle Ax, x \rangle \geq 0$ for all $x \in D(A)$ or equivalently if $\| (\alpha + A)x \| \geq \alpha \| x \|$ for all $x \in D(A)$ and $\alpha > 0$ (Benharrat, 2020).

Definition 2.5: Parallelogram Law

Let a and b be any two elements of the inner product space X. Then,
$\| a + b \|^2 + \| a - b \|^2 = 2(\| a \|^2 + \| b \|^2)\ or \ \| a + b \|^2 \leq 2(\| a \|^2 + \| b \|^2)$ (Kreyszig, 1991).

Definition 2.6: Let $T: H^K \longrightarrow X$ be a continuous linear operator from a Hilbert space H^k into a normed space X. Then$T: H^k \longrightarrow X$ is bounded if there exists a K such that $\| T_x \|_X \leq K \| x \|_{H^k} \ \forall x \in \ H^k$(Kreyszig,, 1991).

Remark 2.1: If the map $T: H^k \longrightarrow X$ is a continuous linear operator from a Hilbert space H^K into another normed space space X, then there must be a continuous extension of $T \ to \ all \ of$ H (Teschl, 2019).

In the section that follows, we will present the existence of a common fixed point of (1.1) which is a nonhomogeneous second order difference equation of accretive type in Hilbert spaces using a contraction and rational fixed point in Hilbert spaces.

RESULTS AND DISCUSSION

Existence of Fixed Points in the Nonhomogeneous Second Order Difference Equation of Accretive Type in Hilbert Spaces.

Lemma 3.1.1: Let A (from (1.1)) be a maximal monotone operator in a Hilbert space $H$with $A^{-1}(0) \neq \emptyset$. Suppose that $\theta_m \geq 0$ and $c_m \geq 0$ and f_m is a sequence in H. Then, there can be more than one mapping in H.

Proof: We let H be in the product space H^k and denote H^k the product space containing k-tupple $u = (u_1 \ldots u_k) \in D(A)^k$ for all $1 \leq i \leq N$ (Apreutesei, 2003). We also denote by $A \subset H^k$ the operator $AU = (c_1 v_1 \ldots c_k v_k), v_i \in A\, u_i, 1 \leq i \leq k + (\theta_1, u, 0 \ldots ,0, x)$ where $u = (u_1 \ldots u_k) \in D(A)^k$ and by $L: H^k \longrightarrow H^k$ the operator, $LU = \big((1 + \theta_1)u_1 - u_2 - \theta_1 u_1 + (1 + \theta_2)u_2 - u_2 \ldots - \theta_{k-1} u_{k-2} + (1 + \theta_{k-1})u_{k-1} - u_k, -\theta_k u_{k-1} + (1 + \theta_k)u_k\big)$.
Therefore, since the operator A is m-accretive in H and the L is continuous everywhere defined and strong, is thus reasonable to conclude that A and L are both continuous mappings on H into itself. Similarly, we can have a third operator $T \in H^K$ on H.
The following Lemma 3.1.2 and its proof show for the uniqueness of space H^K for it to be continuous.

Lemma 3.1.2: Suppose that H^k is dense in H(being a Hilbert space) and that the embedding $H^k \hookrightarrow H$ is continuous. If $T: H^k \longrightarrow X$ is also continuous and X is a Banach space, then there exists a unique continuous linear extension $\tilde{T}: H \longrightarrow X$such that $\tilde{T} \setminus_{H^K} = H$.
Proof: Given that H is a Hilbert space with $H^K \subset H$. We have to show the uniqueness of the space H^K. Let $T_1 , T_2: H \longrightarrow X$ both be continuous (and bounded, see Definition 2.6). Since H^K is dense in H, there exists a sequence $\{x_n\} \subset H^K$ with $x_n \longrightarrow x \epsilon H$. But because on

H^K, they agree with $T, T_1x_n = T_2x_n = Tx_n$ for each n. Thus, we pass limits for uniqueness:
$T_1x = \lim_{n\to\infty} Tx_n = T_2x$. Hence, $T_1x = T_2x$ for every $x \in H$ and so, $T_1 = T_2$. This proves the uniqueness of the space $H^K \subset H$ for it to be continuous.
There is the need to show for uniqueness of the space H^k for it to be continuous.

Theorem 3.1: Let A (from (1.1) be a maximal monotone operator in a Hilbert space H with $A^{-1}(0) \neq \emptyset$. Suppose that $\theta_m \geq 0,\ c_m \geq 0\ and\ f_m$ are sequences in H. Then, equation (1.1) has a solution.

Proof: Let A, L and T be continuous mappings on H(as shown on Lemma 3.1.1) . We have to show that $A(U) \subset L(U), T(U) \subset L(U), AL = AT$ and $TL = LT$ provided $\| Au - Lu \|^2 \leq \alpha(\frac{\|Tu-Tv\|^2+\{\|Tu-Lv\|^2+\|Tv-Au\|^2\}}{\|Tu-Av\|^2+\|Tv-Tv\|^2} + \beta\{\| Tu - Au \|^2 + \| Tv - Lv \|^2\} + \gamma\{\| Tv - Lv \|^2 + \| Tv - Au \|^2\}$, then T, A and L have a common fixed point when $2\alpha + 2\beta + 4\gamma < 1$ for all $u, v \in H$ and$\alpha, \beta, \gamma \in \mathbb{R}$.
Also, let $Au_{2i} = Tu_{2i+1} = u_{2i+2}$ and $Lu_{i+1} = Tu_{2i+2} = u_{2i+3}$.
Taking $u = u_{2i},\ v = u_{2i+1},\ (\mathbf{3.1})$
then, we have

$$\| u_{2i+2} - u_{2i+3} \|^2$$
$$\leq \alpha \frac{\| u_{2i=1} + u_{2i+2} \|^2 (\| u_{2i+1} - u_{2i+3} \|^2 + \| u_{2i+2} - u_{2i+2} \|^2)}{\| u_{2i+1} - u_{2i+2} \|^2 + \| u_{2i+2} + u_{2i+3} \|^2}$$
$$+ \beta\{ \| u_{2i+1} - u_{2i+2} \|^2 + \| u_{2i+2} - u_{2i+3} \|^2\} + \gamma\{$$
$$\| u_{2i+1} - u_{2i+3} \|^2 + \| u_{2i+2} - u_{2i+2} \|^2$$
$$\leq \alpha \frac{\| u_{2i+1} - u_{2i} \|^2\, 2(\| u_{2i+1} - u_{2i+2} \|^2 + \| u_{2i+2} - u_{2i+3} \|^2)}{\| u_{2i+1} - u_{2i+2} \|^2 + \| u_{2i++2} - u_{2i+3} \|^2}$$
$$+ \beta\{\| u_{2i+1} - u_{2i+2} \|^2 + \| u_{2i+2} - u_{2i+3} \|^2\}$$
$$+ 2\{\| u_{2i+1} - u_{2i+2} \|^2 +$$
$$\| u_{2i+2} - u_{2i+3} \|^2\}(By\ parallelogram\ law)\ (\mathbf{3.2})$$

Hence, (u_i) is a Cauchy sequence on H, but H is a Hilbert space. So (u_i) is convergent. Then A, L and T have subsequences. There, they have a common fixed point u, which is the limit point. We will now prove that the sequence (u_i) is Cauchy from equation (3.1). Therefore, there exists a point$u \in H$ such that $(u_i) \longrightarrow u$as $i \longrightarrow \infty$.. Since$A,\ L$ and T are continuous

$$u = \lim_{i\to\infty} u_{2i} = \lim_{i\to\infty} A\,u_{2i+1} = A \lim_{i\to\infty} u_{2i+1}$$
$$= Au\ (that\ is\ u_{2i+1} \longrightarrow u\ as\ i \longrightarrow \infty)(\mathbf{3.3})$$

Also,

$$u = \lim_{i\to\infty} u_{2i} = \lim_{i\to\infty} Lu_{2i+1} = L \lim_{i\to\infty} u_{2i+1}$$
$$= Lu\ (that\ is\ u_{2i+1} \longrightarrow u\ as\ i \longrightarrow \infty)(\mathbf{3.4})$$

Similarly,

$$u = \lim_{i\to\infty} u_{2i} = \lim_{i\to\infty} Tu_{2i+1} = T \lim_{i\to\infty} u_{2i+1}$$
$$= Tu\ (that\ is\ u_{2i+1} \longrightarrow u\ as\ i \longrightarrow \infty)(\mathbf{4.5})$$

From equations (3.3), (3.4) and (3.5), we find a fixed point u which is a common fixed point of A, L and Tin the Hilbert space H.

Theorem 3.2: If from (1.1) a self-mapping in H, that satisfies $\| u - Au \|^2 + \| v - Av \|^2 + \| u - Av \|^2 + \| v - Au \|^2 \leq \frac{\|v-Av\|^2(1+\|u-Au\|^2)}{1+\|u-v\|^2} + \frac{\|u-Au\|^2(1+\|v-Av\|^2)}{1+\|Au-Av\|^2} + \frac{\|u-Av\|^2(1+\|v-Au\|^2)}{1+\|Au-v\|^2} + \frac{\|v-Au\|^2(1+\|u-v\|^2)}{1+\|u-Au\|^2}$. Then, A has a unique fixed point.

Proof: Suppose that $u = u_{2m,} v = u_{2m+1}$ and $Au_{2m} {=} Av_{2m+1}$, then we can have

$$\| u_{2m} - u_{2m+1} \|^2 + \| u_{2m+1} - u_{2m+2} \|^2 +$$
$$\| u_{2m} - u_{2m+2} \|^2 +$$
$$\| u_{2m+1} - u_{2m+1} \|^2$$

Or

$$\| u_{2m} - u_{2m+2} \|^2 + \| u_{2m+1} - u_{2m+2} \|^2 + 2$$
$$\| u_{2m} - u_{2m+1} \|^2 + 2$$
$$\| u_{2m+1} - u_{2m+2} \|^2$$

$\leq \| u_{2m+1} - u_{2m+2} \|^2 + \| u_{2m} - u_{2m+1} \|^2 + 2 \| u_{2m} - u_{2m+1} \|^2 + 2 \| u_{2m+1} - u_{2m+2} \|^2$ (by the parallelogram law). We thus see a contradiction or an absurdity of result. Then,
$\| u_{2m+1} - u_{2m+2} \|^2 = 0$ or $\| u_{2m+1} - u_{2m+2} \| = 0$
Now, suppose that $u = u_{2m+1}$ and $v = u_{2m+2}$ as claimed above, we have

$$\| u - v \| = 0 \ \Longrightarrow u - v \Longrightarrow u = v$$

This signifies uniqueness of the result.

CONCLUSION

This study has established the existence of a common fixed point as well as its uniqueness in a nonhomogeneous second order difference equation (1.1) of accretive type in Hilbert spaces.

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