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ON EXACT FINITE DIFFERENCE SCHEME FOR THE COMPUTATION OF SECOND-ORDER FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT

A special form of Non-Standard Finite Difference Method (NSFDM) called the Exact Finite Difference Scheme (EFDS) for the computation of second-order Fredholm Integro-differential equation shall be constructed in this research. In carrying out the construction of the method, it was assumed that at any point within the interval of integration, the approximate/numerical solution coincides with the exact/theoretical solution. The analysis of the method was also carried out to show that the second-order Fredholm integro-differential equation that possess solutions also have their corresponding EFDS. The method derived was then applied on some modeled second-order Fredholm integro-differential equations and from the results obtained, it is obvious that the EFDS derived did not exhibit any numerical instabilities. As a matter of fact, the computed solutions of the EFDS are exactly equal to the exact solutions.

Keywords: *EFDS, Fredholm equations, integro-differential equations, NSFDM, second-order*

INTRODUCTION

Introduced by Volterra for the first time in the early 1900s, the integro-differential equation is one of the most applied equations in science and engineering. It is an equation that involves both integrals and derivatives of a function. Volterra investigated the population growth, focusing his study on the hereditary influences; where through his research work the topic of integro-differential equations was established, Wazwaz (2011).

It is important to note that in the integro-differential equations, the unknown function $y(x)$ and one or more of its derivatives such as $y'(x)$, $y''(x)$ ... appear out and under the integral sign as well, Mohammed *et. al.,* (2016). It can be classified into Fredholm equations and Volterra equations. The upper bound of the region for integral part of Volterra type is a variable while it is a fixed number for that of Fredholm type.

In this paper, we consider the numerical solution of second order Fredholm integro-differential equations of the form,

$$y''(x) = f(x,y) + \int_a^b k(x,t)y(t)dt \;, \quad a \le x \le b \qquad ......(1)$$

subject to the initial conditions

$$y(a) = \alpha, \;\; y'(a) = \beta \qquad ....(2)$$

where α *and* β are real constants. The function $f(x,y)$ and the kernel $k(x,t)$ are known. The solution $y(x)$ is to be determined. We assume that the problem (1) is well-posed; that is, the problem has the following properties;

- a solution exists,
- the solution is unique, and
- the solution's behavior changes continuously with the initial conditions

Fredholm integro-differential equations model many situations in science and engineering, such as in circuit analysis. The activity of interacting inhibitory and excitatory neurons can be described by a system of integro-differential equations. They are also of significant importance in modeling numerous physical processes such as signal processing and neural networks (Kanwal, 1997). The applications of Fredholm integro-differential equations in electromagnetic theory and dispersive waves and ocean circulations are enormous, Mohammed *et al.,* (2016).

The EFDS is a special form of Non-Standard Finite Difference Method (NSFDM). The exact discretization technique method was first discussed by Potts (1982). He considered the question that whether a linear ordinary difference equation that has the same general solution with the given linear ordinary differential equation can be determined. Also, according to Agarwal (2000), any ordinary differential equation has the exact discretization if its solution exists.

Many authors have developed different methods for the solution of problems of the form (1). These methods include Lagrange interpolation method (Rashed, 2004), Tau operational method (Mohammad and Shahmorad, 2005), Legendre polynomial method (Saadatmandi and Dehghan, 2010), generalized minimal residual method (Aruchuman and Sulaiman, 2010), differential transform method with Adomian polynomials (Behiry, 2013), canonical basis function method (Taiwo, Ganiyu and Okperhie, 2014), power series and chebyshev series approximation methods (Gegele, Evans and Akoh, 2014), Bessel function method (Parand and Nikarya, 2014), cubic spline collocation method (Taiwo and Gegele, 2014), nonstandard finite difference method (Pandey, 2015), homotopy analysis transform method (Mohammed *et al.,* 2016), among others. However, in this research we shall apply an EFDS for the computation of Fredholm integro-differential equations of the form (1).

Definition 1: (Anguelov and Lubuma, 2001)

A finite difference scheme is called non-standard finite difference method, if at least one of the following conditions is met;

i) in the discrete derivative, the traditional denominator is replaced by a non-negative function such that,

$$\phi(h) = h + o(h^2), \; as\, h \to 0 \qquad ....(3)$$

ii) non-linear terms that occur in the differential equation are approximated in a non-local way i.e. by a suitable functions of several points of the mesh. For example,

$$\left.\begin{aligned} y^2 &\approx y_n y_{n+1}, \; y_{n-1}y_n \\ y^3 &\approx y_{n-1}y_n y_{n+1}, \; y_n^2 y_{n+1} \end{aligned}\right\} \qquad ....(4)$$

Definition 2: (Mickens, 1994)

An EFDS is one for which the solution to the difference equation has the same general solution as the associated differential equation.

Below, we give the standard finite discrete representations for some derivatives;

$$\frac{dy}{dx} \to \frac{y_{n+1} - y_n}{h} \qquad ....(5)$$

$$\frac{d^2y}{dx^2} \to \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \qquad ....(6)$$

MATERIALS AND METHODS

Analysis of the Exact Finite Difference Scheme

In carrying out the analysis of the EFDS, it is important to state that the solution to equation (1) can be written as,

$$y(t) = \phi(\lambda, y_0, y_0', t_0, t) \quad ....(7)$$

with

$$\left.\begin{aligned} \phi(\lambda, y_0, t_0, t_0) &= y_0 \\ \phi(\lambda, y_0', t_0, t_0) &= y_0' \end{aligned}\right\} \quad ....(8)$$

Consider a discrete model of equation (1) given by,

$$y_{n+1} = g(\lambda, h, y_n, y_{n-1}, t_n),\ t_n = \boldsymbol{h} \quad ....(9)$$

Its solution can be expressed in the form,

$$y_n = \psi(\lambda, h, y_0, y_0', t_0, t_n) \quad ....(10)$$

with

$$\left.\begin{aligned} \psi(\lambda, y_0, t_0, t_0) &= y_0 \\ \psi(\lambda, y_0', t_0, t_0) &= y_0' \end{aligned}\right\} \quad ....(11)$$

Definition 3

Equations (1) and (9) are said to have the same general solution if and only if

$$y_n = y(t_n) \quad ....(12)$$

for arbitrary values of h .

Theorem 1

The differential equation (1) has an EFDS given by the expression,

$$y_{n+1} = \phi[\lambda, y_n, y_{n-1}, t_{n-1}, t_n, t_{n+1}] \quad ....(13)$$

where ϕ is that of equation (7).

Proof

The group property of the solutions to equation (1) gives,

$$y(t+h) = \phi[\lambda, y(t)\ t-h, t, t+h] \quad ....(14)$$

If we now make the modifications,

$$t \to t_n,\ \ y(t) \to y_n \quad ....(15)$$

then, equation (14) becomes,

$$y_{n+1} = \phi[\lambda, y_n, y_{n-1} t_{n-1}, t_n, t_{n+1}] \quad ....(16)$$

This is the required ordinary difference equation that has the same general solution as equation (1). It is important to note the following implications from the theorem above.

(i) If all solutions of (1) exist for all time, $T = \infty$, then equation (14) holds for all *t* and h .
Otherwise, the relation is assumed to hold whenever the right side of (14) is well-defined

(ii) The theorem is only an existence theorem. That is, if an ordinary differential equation has a solution, then an EFDS exists. According to Mickens (1994), no guidance is given as to how to actually construct such a scheme.

(iii) A major implication of the theorem is that the solution of the difference equation is exactly equal to the solution of the ordinary differential equation on the computational grid for fixed, but, arbitrary step-size .

Formulation of the New Exact Finite Difference Scheme

Theorem 1 stated earlier shall be adopted in constructing a new EFDS for second-order Fredholm integro-differential equations of the form (1) for which exact general solutions are explicitly known. This scheme has the property that its solutions do not have numerical instabilities.

It is important however to note that given a set of linearly independent functions,

$$\{y^i(t)\}, i = 1,2,...,N \quad ....(17)$$

It is always possible to construct an *Nth* order linear difference equation that has the corresponding discrete functions as solutions (Mickens, 1990). Let,

$$y_n^{(i)} \equiv y^{(i)}(t_n),\ t_n = (\Delta t)n = hn \quad ....(18)$$

Then the following determinant gives the required difference equation,

$$\begin{vmatrix} y_n & y_n^{(1)} & y_n^{(2)} & \cdots & y_n^{(k)} \\ y_{n+1} & y_{n+1}^{(1)} & y_{n+1}^{(2)} & \cdots & y_{n+1}^{(k)} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ y_{n+k} & y_{n+k}^{(1)} & y_{n+k}^{(2)} & \cdots & y_{n+k}^{(k)} \end{vmatrix} = 0 \quad ....(19)$$

Consider the equation of the form (1), let us assume that the exact solution of the problem (1) at the point $t = t_n$ denoted by $y(t_n)$ has the same general solution with the numerical solution of the difference equation at the same point $t = t_n$ denoted by y_n. Thus, from equation (19), the corresponding difference equation is given by,

$$\begin{vmatrix} y_n & y_n^{(1)} & y_n^{(2)} \\ y_{n+1} & y_{n+1}^{(1)} & y_{n+1}^{(2)} \\ y_{n+2} & y_{n+2}^{(1)} & y_{n+2}^{(2)} \end{vmatrix} = \begin{vmatrix} y_n & y(t_n) & y'(t_n) \\ y_{n+1} & y(t_{n+1}) & y'(t_{n+1}) \\ y_{n+2} & y(t_{n+2}) & y'(t_{n+2}) \end{vmatrix} = 0 \quad ...(20)$$

Evaluating the determinant of (20), we obtain

$$\begin{aligned} &y_{n+2}\big(y(t_n)y'(t_{n+1}) - y'(t_n)y(t_{n+1})\big) + y_{n+1}\big(y'(t_n)y(t_{n+2}) - y'(t_{n+2})y(t_n)\big) \\ &+ y_n\big(y'(t_{n+2})y(t_{n+1}) - y'(t_{n+1})y(t_{n+2})\big) = 0 \end{aligned} \quad ..(21)$$

Solving (21) for , we obtain

$$y_{n+2} = -\frac{\begin{bmatrix} y_{n+1}\big(y'(t_n)y(t_{n+2}) - y'(t_{n+2})y(t_n)\big) \\ + y_n\big(y'(t_{n+2})y(t_{n+1}) - y'(t_{n+1})y(t_{n+2})\big) \end{bmatrix}}{\big(y(t_n)y'(t_{n+1}) - y'(t_n)y(t_{n+1})\big)} \quad ......(22)$$

Shifting downward the index by one unit, we obtain

$$y_{n+1} = -\frac{\begin{bmatrix} y_n\big(y'(t_{n-1})y(t_{n+1}) - y'(t_{n+1})y(t_{n-1})\big) \\ + y_{n-1}\big(y'(t_{n+1})y(t_n) - y'(t_n)y(t_{n+1})\big) \end{bmatrix}}{\big(y(t_{n-1})y'(t_n) - y'(t_{n-1})y(t_n)\big)} \quad ....(23)$$

Equation (23) is the EFDS capable of solving any problem of the form (1). It is important to note that the EFDS (23) is of the form (13)

RESULTS AND DISCUSSION

Numerical Experiments

The EFDS developed in this research shall be adopted in solving some modeled real-life Fredholm integro-differential equations of the form (1). The following notations shall be used in the tables below.

EMGAA - Absolute error in Mohammed *et. al*., (2016)

EGEA - Absolute error in Gegele *et. al.,* (2014)

ETG - Absolute error in Taiwo and Gegele (2014)

Exec t/sec. - Execution time per seconds for computation at each stage

Problem 1:

Consider the model Fredholm integro-differential equation,

$$y''(x) = e^x - x + \int_0^1 xty(t)dt, \quad 0 \le x \le 1 \quad .....(24)$$

subject to the initial conditions,

$$y(0) = 1,\ y'(0) = 1 \quad ....(25)$$

The exact solution to the problem is given by,

$$y(x) = e^x \quad ....(26)$$

Source: Mohammed *et. al.,* (2016)

On the application of the newly derived EFDS (23) on Problem 1 we obtain the result presented in Table 1 below.

Table 1: Showing the result for Problem 1

x	Exact Solution	Computed Solution	Error	EMGAA	Exec.
0.1000	1.1051709180756477	1.1051709180756477	0.000000e+000	2.01e-008	0.0294
0.2000	1.2214027581601699	1.2214027581601699	0.000000e+000	1.27e-008	0.0396
0.3000	1.3498588075760032	1.3498588075760032	0.000000e+000	1.36e-007	0.0463
0.4000	1.4918246976412703	1.4918246976412703	0.000000e+000	5.25e-007	0.0568
0.5000	1.6487212707001282	1.6487212707001282	0.000000e+000	2.29e-006	0.0664
0.6000	1.8221188003905091	1.8221188003905091	0.000000e+000	3.98e-006	0.0667
0.7000	2.0137527074704766	2.0137527074704766	0.000000e+000	1.59e-005	0.0668
0.8000	2.2255409284924679	2.2255409284924679	0.000000e+000	7.76e-004	0.0670
0.9000	2.4596031111569499	2.4596031111569499	0.000000e+000	3.67e-004	0.0671
1.0000	2.7182818284590455	2.7182818284590455	0.000000e+000	5.65e-004	0.0672

Problem 2:
Consider the model Fredholm integro-differential equation,

$$y''(x) = 32x + \int_{-1}^{1}(1 - xt)y(t)dt, \quad -1 \le x \le 1 \text{............(27)}$$

subject to the initial conditions,

$$y(0) = 1, \quad y'(0) = 1 \quad \text{...................................(28)}$$

The exact solution to the problem is given by,

$$y(x) = 1 + \frac{3}{2}x^2 + 5x^3 \quad \text{..............................(29)}$$

Source: Gegele *et. al.,* (2014)

On the application of the newly derived EFDS (23) on Problem 2 we obtain the result presented in Table 2 below.

Table 2: Showing the result for Problem 2

x	Exact Solution	Computed Solution	Error	EGEA	Exec.
0.1000	1.0199999999999998	1.0199999999999998	0.000000e+000	1.250e-005	0.0136
0.2000	1.1000000000000001	1.1000000000000001	0.000000e+000	5.000e-004	0.0215
0.3000	1.2700000000000000	1.2700000000000000	0.000000e+000	4.375e-004	0.0218
0.4000	1.5600000000000001	1.5600000000000001	0.000000e+000	3.700e-004	0.0220
0.5000	2.0000000000000000	2.0000000000000000	0.000000e+000	2.625e-004	0.0223
0.6000	2.6200000000000006	2.6200000000000006	0.000000e+000	2.000e-004	0.0225
0.7000	3.4500000000000002	3.4500000000000002	0.000000e+000	1.875e-004	0.0226
0.8000	4.5200000000000005	4.5200000000000005	0.000000e+000	1.300e-003	0.0317
0.9000	5.8600000000000003	5.8600000000000003	0.000000e+000	1.212e-003	0.0319
1.0000	7.5000000000000000	7.5000000000000000	0.000000e+000	2.500e-003	0.0321

Problem 3:
Consider the model Fredholm integro-differential equation,

$$y''(x) = \frac{5}{3} - 11x + \int_{0}^{1} y(t)dt, \quad 0 \le x \le 1 \quad \text{........................(30)}$$

subject to the initial conditions,

$$y(0) = y'(0) = 1 \quad \text{...(31)}$$

The exact solution to the problem is given by,

$$y(x) = 1 + x + \frac{5}{6}x^2 - \frac{5}{3}x^3 \qquad (32)$$

Source: Taiwo and Gegele (2014)

On the application of the newly derived EFDS (23) on Problem 3 we obtain the result presented in Table 3 below.

Table 3: Showing the result for Problem 3

	Exact Solution	Computed Solution	Error	ETG	Exec.
0.1000	1.1066666666666667	1.1066666666666667	0.000000e+000	3.489e-006	0.0116
0.2000	1.2200000000000000	1.2200000000000000	0.000000e+000	3.410e-006	0.0231
0.3000	1.3300000000000001	1.3300000000000001	0.000000e+000	2.983e-006	0.0236
0.4000	1.4266666666666665	1.4266666666666665	0.000000e+000	2.837e-006	0.0237
0.5000	1.5000000000000000	1.5000000000000000	0.000000e+000	2.602e-006	0.0241
0.6000	1.5400000000000000	1.5400000000000000	0.000000e+000	2.591e-006	0.0242
0.7000	1.5366666666666666	1.5366666666666666	0.000000e+000	2.429e-006	0.0243
0.8000	1.4800000000000000	1.4800000000000000	0.000000e+000	1.994e-006	0.0244
0.9000	1.3599999999999999	1.3599999999999999	0.000000e+000	1.405e-006	0.0245
1.0000	1.1666666666666667	1.1666666666666667	0.000000e+000	1.067e-008	0.0247

Problem 4:

Consider the model Fredholm integro-differential equation,

$$y''(x) = 10 - \frac{146}{35}x + \frac{1}{2}\int_{-1}^{1} xty^2(t)dt, \quad -1 \le x \le 1 \qquad \text{.......................(33)}$$

subject to the initial conditions,

$$y(0) = 1, \; y'(0) = 0 \qquad \text{..(34)}$$

The exact solution to the problem is given by,

$$y(x) = 1 + 5x^2 - x^3 \qquad \text{..(35)}$$

Source: Taiwo and Gegele (2014)

On the application of the newly derived EFDS (23) on Problem 4 we obtain the result presented in Table 4 below.

Table 4: Showing the result for Problem 4

x	Exact Solution	Computed Solution	Error	ETG	Exec.
0.1000	1.0490000000000002	1.0490000000000002	0.000000e+000	6.008e-008	0.0124
0.2000	1.1919999999999999	1.1919999999999999	0.000000e+000	7.918e-008	0.0218
0.3000	1.4230000000000003	1.4230000000000003	0.000000e+000	8.432e-007	0.0312
0.4000	1.7360000000000002	1.7360000000000002	0.000000e+000	6.884e-007	0.0316
0.5000	2.1250000000000000	2.1250000000000000	0.000000e+000	5.718e-007	0.0320
0.6000	2.5840000000000005	2.5840000000000005	0.000000e+000	5.623e-007	0.0321
0.7000	3.1070000000000007	3.1070000000000007	0.000000e+000	4.009e-007	0.0322
0.8000	3.6880000000000011	3.6880000000000011	0.000000e+000	2.929e-007	0.0323
0.9000	4.3210000000000006	4.3210000000000006	0.000000e+000	2.887e-007	0.0325
1.0000	5.0000000000000000	5.0000000000000000	0.000000e+000	1.999e-007	0.0326

From the results generated in Tables 1-4, it is clear that the EFDS in equation (23) is computationally reliable and efficient. This is because the computed solution matches exactly with the exact solution for each of the problems. It is also obvious from the results that the method performed better than the ones with which we compared our results with. The method is also efficient because from the tables, the execution times per seconds are very small. This shows that the method generates results very fast.

CONCLUSION

A new EFDS has been developed in this paper for the solution of second order Fredholm Integro-differential equations of the form (1). The method developed was applied on some modeled problems and from the results obtained, it is clear that the method is computationally reliable. The analysis of the method derived was also carried out. A major advantage of the method is that it does not exhibit any numerical instability.

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